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A multiplicity theorem for problems with the p-Laplacian

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Abstract

We consider a nonlinear elliptic problem driven by the p-Laplacian, with a parameter $\lambda \in \mathbb{R}$ and a nonlinearity exhibiting a superlinear behavior both at zero and at infinity. We show that if the parameter λ is bigger than $\lambda_2 =$ the second eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$, then the problem has at least three nontrivial solutions. Our approach combines the method of upper–lower solutions with variational techniques involving the Second Deformation Theorem. The multiplicity result that we prove extends an earlier semilinear (i.e. $p = 2$) result due to Struwe [M. Struwe, Variational Methods, Springer-Verlag, Berlin, 1990].

Keywords: Multiple nontrivial solutions; Superlinear nonlinearity; Upper and lower solutions; Eigenvalues of the p-Laplacian; Second deformation theorem

1. Introduction

In this paper we prove a multiplicity theorem for nonlinear elliptic problems driven by the p-Laplacian. So suppose $Z \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary ∂Z. The problem under consideration is the following:

$$\left\{ \begin{array}{ll} -\operatorname{div}\big(\|Dx(z)\|^{p-2} Dx(z)\big) = \lambda |x(z)|^{p-2} x(z) - f\big(z, x(z)\big) & \text{a.e. on } Z \\ x|_{\partial Z} = 0, \quad \lambda \in \mathbb{R},\ 1 < p < \infty. & \end{array} \right\} \tag{1.1}$$

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We are interested on multiplicity results when the nonlinearity $f(z,x)$ exhibits a "superlinear" behavior both at zero and at $\pm\infty$. In the past this problem was investigated in the case $p=2$ (semilinear case). First, Ambrosetti, Mancini [2] proved that if $\lambda>\lambda_1$ (λ_1 being the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$), then the problem has two nontrivial solutions of constant sign (one positive and the other negative). Soon thereafter Struwe [11] improved the result and proved that if $\lambda>\lambda_2$ the problem (1.1) has three nontrivial solutions. Subsequently Ambrosetti, Lupo [1] slightly improved the work of Struwe [11] and also presented an approach based on Morse theory. This, of course, required that the nonlinearity $f(z,\cdot)$ is C^1. The most general result for the semilinear case can be found in Struwe [12, p. 132], who succeeded in eliminating the differentiability condition on the nonlinearity f and simplified the argument of Ambrosetti, Lupo [1]. We remark, however, that still Struwe [12] requires that the nonlinearity f (which he assumes it to be independent of z), is Lipschitz continuous. When $p\neq 2$ (nonlinear problem), we are not aware of any such multiplicity results for problem (1.1). Here we present such a generalization of the result of Struwe [12].

2. Preliminaries

First let us briefly recall some basic facts about the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$. So we consider the following nonlinear eigenvalue problem:

$$\left\{\begin{array}{ll} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big)=\lambda\big|x(z)\big|^{p-2}x(z) & \text{a.e. on } Z \\ x|_{\partial Z}=0, \quad \lambda\in\mathbb{R},\ 1<p<\infty. & \end{array}\right\} \tag{2.1}$$

The least real number λ for which problem (2.1) has a nontrivial solution is called the first eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and it is denoted by λ_1. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the corresponding eigenspace is one-dimensional). There is a variational characterization of $\lambda_1>0$, via the Rayleigh quotient, i.e.

$$\lambda_1=\min\left\{\frac{\|Dx\|_p^p}{\|x\|_p^p}\colon x\in W_0^{1,p}(Z),\ x\neq 0\right\}. \tag{2.2}$$

This minimum is realized at the normalized principal eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so it follows that u_1 does not change sign on Z. Thus we may assume that $u_1\geqslant 0$. Moreover, from the nonlinear regularity theory (see Lieberman [10]), we know that $u_1\in C_0^1(\bar Z)$. In addition, via the nonlinear strict maximum principle of Vazquez [13], we have that $u_1(z)>0$ for all $z\in Z$ and $\frac{\partial u_1}{\partial n}(z)<0$ for all $z\in\partial Z$. If we consider the ordered Banach space $C_0^1(\bar Z)=\{x\in C^1(\bar Z)\colon x(z)=0 \text{ for all } z\in\partial Z\}$ with positive cone

$$C_0^1(\bar Z)_+=\big\{x\in C_0^1(\bar Z)\colon x(z)\geqslant 0 \text{ for all } z\in\bar Z\big\},$$

we know that $\operatorname{int} C_0^1(\bar Z)_+\neq\emptyset$ and is given by

$$\operatorname{int} C_0^1(\bar Z)_+=\left\{x\in C_0^1(\bar Z)_+\colon x(z)>0 \text{ for all } z\in Z \text{ and } \frac{\partial x}{\partial n}(z)<0 \text{ for all } z\in\partial Z\right\}.$$

Therefore $u_1\in\operatorname{int} C_0^1(\bar Z)_+$.