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Piecewise rigidity

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Abstract

In this paper we provide a Liouville type theorem in the framework of fracture mechanics, and more precisely in the theory of *SBV* deformations for cracked bodies. We prove the following rigidity result: if $u \in SBV(\Omega, \mathbb{R}^N)$ is a deformation of Ω whose associated crack J_u has finite energy in the sense of Griffith's theory (i.e., $\mathcal{H}^{N-1}(J_u) < \infty$), and whose approximate gradient ∇u is almost everywhere a rotation, then u is a collection of an at most countable family of rigid motions. In other words, the cracked body does not store elastic energy if and only if all its connected components are deformed through rigid motions. In particular, global rigidity can fail only if the crack disconnects the body.

Keywords: Rigidity; Liouville theorem; *SBV* functions; Discontinuities; Caccioppoli partitions

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1. Introduction

A classical rigidity result in nonlinear elasticity, due to Liouville, states that if an elastic body is deformed in such a way that its deformation gradient is pointwise a rotation, then the body is indeed subject to a rigid motion. If the body is supposed to be hyperelastic with an elastic energy density $\mathcal{W}$ defined on a *natural* reference configuration Ω, a standard assumption for $\mathcal{W}$ which comes from its *frame indifference* is that $\mathcal{W}$ is minimized exactly on the set of rotations SO(3). Hence the rigidity result implies that the body does not store elastic energy if and only if it is deformed through a rigid motion.

From a mathematical viewpoint, Liouville's theorem can be stated as follows: if $\Omega \subseteq \mathbb{R}^N$ is open and connected, $u \in C^\infty(\Omega; \mathbb{R}^N)$ is such that $\nabla u(x) \in \mathrm{SO}(N)$ for every $x \in \Omega$, then $u = Rx + b$ for some $b \in \mathbb{R}$ and $R \in \mathrm{SO}(N)$. The assumption on the regularity of u has been fairly weakened, and now the same rigidity result is available for deformations in the class of Sobolev maps (see Yu. Reshetnyak [19]). In this case the deformation gradient is defined only almost everywhere in Ω, so that the assumption for rigidity is $\nabla u(x) \in \mathrm{SO}(N)$ for a.e. $x \in \Omega$.

A quantitative rigidity estimate has been provided recently by Friesecke, James and Müller [14], in order to derive nonlinear plates theories from three-dimensional elasticity. They proved that if Ω is connected and with Lipschitz boundary, there exists a constant C depending only on Ω and N such that for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$

$$\min_{R \in \mathrm{SO}(N)} \|\nabla u - R\|_{L^2(\Omega)} \leqslant C \big\| \mathrm{dist}\big(\nabla u, \mathrm{SO}(N)\big) \big\|_{L^2(\Omega)}. \tag{1.1}$$

As a consequence, if the deformation gradient is close to rotations (in L^2), then it is in fact close to a unique rotation. Estimate (1.1) is indeed true in L^p for every $1 < p < +\infty$, and this can be proved with minor modification of the arguments of [14].

The aim of this paper is to discuss the problem of rigidity in the framework of fracture mechanics, that is for bodies that can not only deform elastically, but also be cracked along surfaces where the deformation becomes discontinuous. The class of admissible deformations that we consider, in this setting, will be the space of *special functions of bounded variation* $SBV(\Omega; \mathbb{R}^N)$ (see Section 2 for a precise definition). Given $u \in SBV(\Omega; \mathbb{R}^N)$, the approximate gradient ∇u (which exists at almost every point of Ω) takes into account the elastic part of the deformation, while the jump set J_u represents a crack in the reference configuration. The set J_u is rectifiable, that is, it can be covered (up to a $\mathcal{H}^{N-1}$-negligible set) by a countable number of C^1 submanifolds of $\mathbb{R}^N$. So J_u is, in some sense, an $(N-1)$-dimensional surface.

In the context of *SBV* deformations, we cannot expect a rigidity result as for elastic deformations, because a crack can divide the body into two parts, each one subject to a different rigid deformation. We prove that this is essentially the only way rigidity can be violated, provided the crack J_u has "finite energy" (which, in the framework of Griffith's theory, means that its total