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The structure of shift–modulation invariant spaces: The rational case

Marcin Bownik [1]

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA

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Abstract

In this paper we study structural properties of shift–modulation invariant (SMI) spaces, also called Gabor subspaces, or Weyl–Heisenberg subspaces, in the case when shift and modulation lattices are rationally dependent. We prove the characterization of SMI spaces in terms of range functions analogous to the well-known description of shift-invariant spaces [C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78; M. Bownik, The structure of shift-invariant subspaces of $L_2(\mathbb{R}^n)$, J. Funct. Anal. 177 (2000) 282–309; H. Helson, Lectures on Invariant Subspaces, Academic Press, New York/London, 1964]. We also give a simple characterization of frames and Riesz sequences in terms on their behavior of the fibers of the range function. Next, we prove several orthogonal decomposition results of SMI spaces into simpler blocks, called principal SMI spaces. Then, this is used to characterize operators invariant under both shifts and modulations in terms of families of linear maps acting on the fibers of the range function. We also introduce the fundamental concept of the dimension function for SMI spaces. As a result, this leads to the classification of unitarily equivalent SMI spaces in terms of their dimension functions. Finally, we show several results illustrating our fiberization techniques to characterize dual Gabor frames.

Keywords: Gabor subspace; Weyl–Heisenberg subspace; Shift-modulation invariant space; Frame sequence; Riesz sequence; Dual frame; Dimension function; Range function; Range operator

E-mail address: mbownik@uoregon.edu.
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1. Introduction

The aim of this paper is to investigate the structure of shift–modulation invariant spaces. These are the subspaces of $L^2(\mathbb{R}^n)$ generated by Gabor systems, also called Weyl–Heisenberg systems. Gabor systems are a subject of the intensive study [4–6,9–17,19]. One of the fundamental problems in this area is to determine when two SMI spaces are unitarily equivalent, i.e., there exists a unitary operator between these spaces commuting both with shifts and modulatCCions. A similar problem in the context of shift-invariant (SI) spaces was settled by the author in [2]. It was proved that two SI spaces are unitarily equivalent if and only if their dimension functions coincide a.e. Recall from [1–3] that the dimension function of a SI space V is a $\mathbb{Z}^n$-periodic function $\dim_V : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$, which measures the dimensions of the fibers of the range function corresponding to V.

The SMI spaces have, in general, a much more complex structure than their SI counterparts, since they must also obey modulation invariance. Obviously, every SMI space is also SI, hence every result about SI spaces can be applied in the shift–modulation setting. This might be a bit misleading, since SMI spaces are a very special kind of SI spaces. In particular, one can easily prove that their SI dimension functions take only two possible values: 0 or ∞. This is a consequence of the fact that every SMI space can be realized as a SI space with an infinite set of generators being modulations of each other. Therefore, general results about SI spaces have a limited applicability in the SMI setting and there is a need to develop a genuine shift–modulation theory.

The main goal of this work is to show that this is indeed possible if modulation and shift lattices are rationally dependent. The case when lattices are not rationally dependent requires a different set of techniques and it will not be treated here. Despite that our theory of SMI spaces is closely parallel to the shift-invariant theory, there are some significant differences setting them apart. To describe our results in some detail we need to recall a basic terminology.

Definition 1.1. Let Λ, Γ be two full rank lattices in $\mathbb{R}^n$, i.e., $\Lambda = P_0\mathbb{Z}^n$, $\Gamma = P_1\mathbb{Z}^n$ for some $n \times n$ non-singular matrices P_0, P_1 with real entries. Let $\mathcal{A} \subset L^2(\mathbb{R}^n)$ be a countable set of generators. The *Gabor system* $G(\mathcal{A}, \Lambda, \Gamma)$ is the set of translation and modulation shifts

$$G(\mathcal{A}, \Lambda, \Gamma) = \{M_\lambda T_\gamma \varphi \colon \lambda \in \Lambda,\ \gamma \in \Gamma,\ \varphi \in \mathcal{A}\}, \tag{1.1}$$

where $M_\lambda f(x) = e^{2\pi i \langle x, \lambda\rangle} f(x)$, $T_\gamma f(x) = f(x - \gamma)$. We say that a closed subspace $V \subset L^2(\mathbb{R}^n)$ is *shift–modulation invariant* (SMI) if

$$M_\lambda T_\gamma V \subset V \quad \text{for all } \lambda \in \Lambda,\ \gamma \in \Gamma. \tag{1.2}$$

The smallest SMI space generated by $\mathcal{A}$ is denoted by

$$S(\mathcal{A}, \Lambda, \Gamma) = \overline{\operatorname{span}} G(\mathcal{A}, \Lambda, \Gamma).$$

We say that two lattices Λ and Γ are *rationally dependent* if $\Lambda \cap \Gamma$ is a full rank lattice. Applying the standard dilation argument one can assume that the modulation lattice $\Lambda = \mathbb{Z}^n$, or alternatively that the shift lattice $\Gamma = \mathbb{Z}^n$. Then, any result involving Gabor systems with general lattices Λ, Γ can be deduced from a corresponding result when one of the lattices is $\mathbb{Z}^n$.