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ScienceDirect
Journal of Functional Analysis 257 (2009) 1261–1287
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Uniqueness results for nonlocal Hamilton–Jacobi equations ☆

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Received 18 March 2008; accepted 19 April 2009

Available online 13 May 2009

Communicated by H. Brezis

Abstract

We are interested in nonlocal eikonal equations describing the evolution of interfaces moving with a nonlocal, non-monotone velocity. For these equations, only the existence of global-in-time weak solutions is available in some particular cases. In this paper, we propose a new approach for proving uniqueness of the solution when the front is expanding. This approach simplifies and extends existing results for dislocation dynamics. It also provides the first uniqueness result for a Fitzhugh–Nagumo system. The key ingredients are some new perimeter estimates for the evolving fronts as well as some uniform interior cone property for these fronts.

Keywords: Nonlocal Hamilton–Jacobi equations; Dislocation dynamics; Fitzhugh–Nagumo system; Nonlocal front propagation; Level-set approach; Geometrical properties; Lower-bound gradient estimate; Viscosity solutions; Eikonal equation; L^1-dependence in time

☆ This work was partially supported by the ANR (Agence Nationale de la Recherche) through MICA project (ANR-06-BLAN-0082).

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doi:10.1016/j.jfa.2009.04.014

1. Introduction

In this article, we are interested in uniqueness results for different types of problems which can be written as nonlocal Hamilton–Jacobi equations of the following form:

$$u_t = c[\mathbb{1}_{\{u \geqslant 0\}}](x,t)|Du| \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $T > 0$, the solution u is a real-valued function, u_t and Du stand respectively for its time and space derivatives and $\mathbb{1}_A$ is the indicator function of a set A. Finally u_0 is a bounded, Lipschitz continuous function.

For any indicator function χ or more generally for any $\chi \in L^\infty$ with $0 \leqslant \chi \leqslant 1$ a.e., the function $c[\chi]$ depends on χ in a nonlocal way and, in the main examples we have in mind, it is obtained from χ through a convolution type procedure (either only in space or in space and time). In particular, in our framework, despite the fact that χ has no regularity neither in x nor in t, $c[\chi]$ will always be Lipschitz continuous in x; on the contrary we impose no regularity with respect to t. More precisely we always assume in what follows that there exist constants C, $\underline{c}$, $\bar{c} > 0$ such that

(H1) For any $\chi \in L^\infty(\mathbb{R}^N \times (0,T), [0,1])$, the velocity $c = c[\chi]$ is (x,t)-measurable and, for all $x, y \in \mathbb{R}^N$ and $t \in [0,T]$,

$$\begin{gathered} \bigl|c(x,t) - c(y,t)\bigr| \leqslant C|x-y|, \\ 0 < \underline{c} \leqslant c(x,t) \leqslant \bar{c}. \end{gathered} \tag{1.3}$$

The first consequence of assumption **(H1)** is that, for any given function $\chi \in L^\infty(\mathbb{R}^N \times (0,T), [0,1])$, there exists a unique solution to

$$\begin{cases} u_t(x,t) = c[\chi](x,t)\bigl|Du(x,t)\bigr| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

for any bounded and Lipschitz continuous initial data u_0.

To give a first flavor of our main uniqueness results, we can point out the following key facts: Eq. (1.1) can be seen as the "level-set approach"-equation associated to the motion of the front $\Gamma_t := \{x\colon\ u(x,t) = 0\}$ with the nonlocal velocity $c[\mathbb{1}_{\{u(\cdot,t) \geqslant 0\}}]$. However, in the non-standard examples we consider, it is not only a nonlocal but also non-monotone "geometrical" equation; by non-monotone we mean that the inclusion principle, which plays a central role in the "level-set approach", does not hold and, therefore, the uniqueness of solutions cannot be proved via standard viscosity solutions methods.

In fact, the few uniqueness results which exist in the literature (see below) rely on L^1 type estimates on the solution; this is natural since one has to connect the continuous function u and the indicator function $\mathbb{1}_{\{u \geqslant 0\}}$. The main estimates concern measures of sets of the type $\{x\colon\ a \leqslant u(x,t) \leqslant b\}$ for a, b close to 0. Whether or not the aforementioned estimate has to be uniform on time, or of integral type, strongly depends on the properties of the convolution kernel. In order to emphasize this fact, we are going to concentrate on two model cases: the first one is a dislocation type equation (see Section 3) in which the kernel belongs to L^∞ while the second one is related