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Optimal cubature formulas on compact homogeneous manifolds

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Abstract

We find lower bounds for the rate of convergence of optimal cubature formulas on sets of differentiable functions on compact homogeneous manifolds of rank I or two-point homogeneous spaces. It is shown that these lower bounds are sharp in the power scale in the case of $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$.

Keywords: Reconstruction; Data points; Polynomial; Volume

1. Introduction

Let (Ω, Σ, η) be a measure space, where Ω is a compact domain in $\mathbb{R}^d$, and $\mathcal{K} \in C(\Omega)$ be a given set of real continuous functions, $f : \Omega \to \mathbb{R}$. Let $\{x_1, \ldots, x_n\} \subset \Omega$ be a fixed set of points. It is natural to approximate the integral

$$\int_{\Omega} f \, d\eta$$

by a cubature formula

$$\sum_{k=1}^{n} \alpha_k f(x_k),$$

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where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and to minimize the error of approximation

$$\kappa_n(\mathcal{K}) := \inf_{\{x_1,\ldots,x_n\}\subset\Omega} \inf_{\{\alpha_1,\ldots,\alpha_n\}\subset\mathbb{R}} \sup_{f\in\mathcal{K}} \left| \int_\Omega f\,d\eta - \sum_{k=1}^{n} \alpha_k f(x_k) \right|. \tag{1}$$

The theory of cubature formulas has a long history and the first references are coming from the unmemorable times. In the modern epoch simple cubature (quadrature) formulas have been constructed by Kepler and Torricelli (1664), Simpson (1743), Newton and Cotes (1722). Different important methods of computing of integrals have been developed by Lagrange, Chebyshev, Bernstein, Krylov, Nikol'skij, Sobolev and many others. In the one-dimensional case, on $\mathbb{S}^1$, the unit circle, it is known that the formula of rectangles

$$\int_{\mathbb{S}^1} f(x)\,dx \approx \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2\pi k}{n}\right)$$

is an optimal on Sobolev classes $W^r_\infty(\mathbb{S}^1) = \{f \mid f^{(r)} \in U_\infty(\mathbb{S}^1)\}$, where $U_\infty(\mathbb{S}^1) = \{\phi \mid \|\phi\|_\infty \leqslant 1\}$ and $r \in \mathbb{N}$. Remark that an analogous result is unknown for fractional values of $r > 0$.

Observe that the extremal problem (1) and their discrete analogs are in spirit of the classical Kolmogorov n-widths of finite-dimensional sets. This range of problems has been extensively studied by Tikhomorov, Makovoz, Kashin, Gluskin and others (see [17] for more details and references).

The problem of numerical integration over the surface of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d \geqslant 2$, is one of the most important in Numerical Analysis and Applications. The theory of functions on $\mathbb{S}^2$ has been initiated in the eighteenth century in works of Laplace and Legendre when the first cubature formulas appeared. Consequently, the problem on an optimal cubature formula on $\mathbb{S}^2$ (in general, $\mathbb{S}^d$, $d \geqslant 2$) remains open since that time. Therefore, the problem on the best cubature formula on $\mathbb{S}^2$ or (in general) on compact Riemannian manifolds $\mathbb{M}^d$ it is natural to call the *Laplace–Legendre* problem.

A fundamental problem in this area is connected with an optimal distribution of data points $x_1, \ldots, x_n$ and finding an optimal coefficients $\alpha_1, \ldots, \alpha_n$ to approximate "well" the integral. Even in the case of $\mathbb{S}^2$, the two-dimensional sphere in $\mathbb{R}^3$, it is not possible to construct, in general, an equidistributed set of data points since there are finitely many polyhedral groups. Different attempts to find sets of points on the sphere which imitate the role of the roots of unity on the unit circle usually led to deep problems of the Geometry of Numbers, Theory of Potential, etc., and usually these approaches give just a measure of a uniform distribution like cup discrepancy or minimum energy configurations.

We consider here optimal cubature formulas for the Sobolev classes $W^r_\infty(\mathbb{M}^d) \subset C(\mathbb{M}^d)$ on a compact two-point homogeneous manifold $\mathbb{M}^d$ defined in Section 2.

The respective extremal problem can be formulated as following. Let $f \in W^r_\infty(\mathbb{M}^d)$ (see Section 2 for the definitions) and $\{x_1, \ldots, x_n\} \subset \mathbb{M}^d$. Consider an information operator $T_n \in \mathcal{L}(C(\mathbb{M}^d), \mathbb{R}^n)$,

$$T_n : C\bigl(\mathbb{M}^d\bigr) \to \mathbb{R}^n,$$
$$f(\cdot) \mapsto \bigl(f(x_1), \ldots, f(x_n)\bigr).$$