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The ultimate estimate of the upper norm bound for the summation of operators

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Abstract

Let A and B be bounded linear operators acting on a Hilbert space H. It is shown that the triangular inequality serves as the ultimate estimate of the upper norm bound for the sum of two operators in the sense that

$$\sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}.$$

Consequences of the result related to spectral sets, the von Neumann inequality, and normal dilations are discussed. Furthermore, it is shown that the above equality can be used to characterize those unitarily invariant norms that are multiples of the operator norm in the finite-dimensional case.

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1. Introduction

Let H be a Hilbert space equipped with the inner product (x, y), and let $B(H)$ be the algebra of bounded linear operators acting on H equipped with the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in H,\ (x, x) = 1\}.$$

If H is n-dimensional, we identify H with $\mathbf{C}^n$ and $B(H)$ with the algebra M_n of $n \times n$ complex matrices.

Basically, the *triangle inequality*

$$\|A + B\| \leqslant \|A\| + \|B\|$$

plays an important role in structure theory concerning the summation of matrices. In spite of the complexity of the norm computation, we will show that there are effective ways to obtain the best norm estimate for the sum of two operators.

For any $A, B \in B(H)$, it is clear that

$$\|U^*AU + V^*BV\| \leqslant \|U^*AU + \mu I\| + \|V^*BV - \mu I\| = \|A + \mu I\| + \|B - \mu I\|$$

for all $\mu \in \mathbf{C}$ and unitary $U, V \in B(H)$. We show that this rather trivial inequality is the ultimate estimate of the upper norm bound for $A + B$ in the sense that

$$\begin{aligned} &\sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ &\quad = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}. \end{aligned} \tag{1.1}$$

The inequality (1.1) is of great significance even when A and B are normal matrices. As established in [3], a sharp bound is obtained for $\|A_1 + iA_2\|$, where A_1 and A_2 are $n \times n$ Hermitian matrices satisfying $b_1 I \leqslant A_1 \leqslant c_1 I$ and $b_2 I \leqslant A_2 \leqslant c_2 I$.

Evidently, if the unitary similarity orbit of $A \in B(H)$ is the collection of operators unitarily similar to A, then the quantity in (1.1) can be viewed as a measure of (or a bound on) the distance between the unitary similarity orbits of A and $-B$. In particular, replacing B by $-B$ and μ by $-\mu$, we can rewrite Eq. (1.1) as

$$\begin{aligned} &\sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ &\quad = \min\{\|A + \mu I\| + \|B + \mu I\| : \mu \in \mathbf{C}\}. \end{aligned}$$

Note that the supremum on the left-hand side of (1.1) may not be attainable in the infinite-dimensional case; see [3, Example 5.1] for the full justification of the following example.