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Free pluriharmonic majorants and commutant lifting ☆

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Abstract

In this paper we initiate the study of sub-pluriharmonic curves and free pluriharmonic majorants on the noncommutative open ball

$$\left[B(\mathcal{H})^n\right]_1 := \left\{(X_1, \dots, X_n) \in B(\mathcal{H})^n \colon \left\|X_1X_1^* + \cdots + X_nX_n^*\right\|^{1/2} < 1\right\},$$

where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Several classical results from complex analysis have analogues in this noncommutative multivariable setting.

We present basic properties for sub-pluriharmonic curves, characterize the class of sub-pluriharmonic curves that admit free pluriharmonic majorants and find, in this case, the least free pluriharmonic majorants. We show that, for any free holomorphic function Θ on $[B(\mathcal{H})^n]_1$, the map

$$\varphi : [0, 1) \to C^*(R_1, \dots, R_n), \quad \varphi(r) := \Theta(rR_1, \dots, rR_n)^*\Theta(rR_1, \dots, rR_n),$$

is a sub-pluriharmonic curve in the Cuntz–Toeplitz algebra generated by the right creation operators $R_1, \dots, R_n$ on the full Fock space with n generators. We prove that Θ is in the noncommutative Hardy space H^2_{ball} if and only if φ has a free pluriharmonic majorant. In this case, we find Herglotz–Riesz and Poisson type representations for the least pluriharmonic majorant of φ. Moreover, we obtain a characterization of the unit ball of H^2_{ball} and provide a parametrization and concrete representations for all free pluriharmonic majorants of φ, when Θ is in the unit ball of H^2_{ball}.

In the second part of this paper, we introduce a generalized noncommutative commutant lifting (GNCL) problem which extends, to our noncommutative multivariable setting, several lifting problems including the classical Sz.-Nagy–Foiaş commutant lifting problem and the extensions obtained by Treil–Volberg, Foiaş–Frazho–Kaashoek, and Biswas–Foiaş–Frazho, as well as their multivariable noncommutative versions. We

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solve the GNCL problem and, using the results regarding sub-pluriharmonic curves and free pluriharmonic majorants on noncommutative balls, we provide a complete description of all solutions. In particular, we obtain a concrete Schur type description of all solutions in the noncommutative commutant lifting theorem.

Keywords: Multivariable operator theory; Noncommutative Hardy space; Fock space; Creation operators; Free holomorphic function; Free pluriharmonic function; Sub-pluriharmonic curves; Commutant lifting

0. Introduction

Noncommutative multivariable operator theory has received a lot of attention, in the last two decades, in the attempt of obtaining a *free* analogue of Sz.-Nagy–Foiaş theory [62], for row contractions, i.e., n-tuples of bounded linear operators $(T_1, \dots, T_n)$ on a Hilbert space such that

$$T_1T_1^* + \cdots + T_nT_n^* \leqslant I.$$

Significant progress has been made regarding noncommutative dilation theory and its applications to interpolation in several variables [1,2,5,8,12,16,19,36–38,41,43,45,47,50–52,54–57], and unitary invariants for n-tuples of operators [3,4,6,38,48,49,51,56].

In [53] and [55], we developed a theory of holomorphic (resp. pluriharmonic) functions in several noncommuting (free) variables and provide a framework for the study of arbitrary n-tuples of operators on a Hilbert space. Several classical results from complex analysis have free analogues in this noncommutative multivariable setting. This theory enhances our program to develop a *free* analogue of Sz.-Nagy–Foiaş theory. In related areas of research, we remark the work of Helton, McCullough, Putinar, and Vinnikov, on symmetric noncommutative polynomials [25–29], and the work of Muhly and Solel on representations of tensor algebras over C^* correspondences (see [31,32]).

The present paper is a natural continuation of [53] and [55]. We initiate here the study of sub-pluriharmonic curves and free pluriharmonic majorants on noncommutative balls. We are led to a characterization of the noncommutative Hardy space H^2_{ball} in terms of free pluriharmonic majorants, and to a Schur type description of the unit ball of H^2_{ball}. These results are used to solve a multivariable commutant lifting problem and provide a description of all solutions.

To put our work in perspective and present our results, we need to set up some notation. Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \dots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$. If $(X_1, \dots, X_n) \in B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on an infinite-dimensional Hilbert space $\mathcal{H}$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$. We recall (see [53, 55]) that a map $f : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ is called free holomorphic function with scalar coefficients if it has a representation

$$f(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha, \quad X := (X_1, \dots, X_n) \in \left[B(\mathcal{H})^n\right]_1,$$