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Schrödinger operators on graphs and geometry I: Essentially bounded potentials

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Abstract

The inverse spectral problem for Schrödinger operators on finite compact metric graphs is investigated. The relations between the spectral asymptotics and geometric properties of the underlying graph are studied. It is proven that the Euler characteristic of the graph can be calculated from the spectrum of the Schrödinger operator in the case of essentially bounded real potentials and standard boundary conditions at the vertices. Several generalizations of the presented results are discussed.

Keywords: Quantum graph; Trace formula; Euler characteristic

1. Introduction

The theory of differential operators on metric graphs is a rapidly developing area of modern mathematical physics. Interest to these problems can be explained not only by important applications in the theory of quantum wave guides and nanoelectronics, but by discovered interesting phenomena putting these problems in the area between ordinary and partial differential operators. Indeed methods originally developed for both areas are successfully applied to study the problems on metric graphs. The main aim of this paper is to study the relation between the spectrum of a Schrödinger operator on such a graph and geometric properties of the graph. This question

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has already been studied for Laplace operators with standard boundary conditions at the vertices (see (1) below) and it was proven that the spectrum of the Laplace operator determines the total length, the number of connected components and the Euler characteristics of the underlying graph (see Definition 3). To establish the relation between the spectrum and the Euler characteristics one used so-called trace formula (see (2)) connecting the spectrum of the Laplacian with the set of periodic orbits on the graph. Using this relation effective formulas for the Euler characteristic can be proven (see (6) and (7)). The main goal of the current paper is to generalize these results to the case of Schrödinger operators on graphs. Such operators are not uniquely determined by the underlying graphs (like Laplacians with standard boundary conditions), but depend on the choice of real valued potential and may be other than the standard boundary conditions at the vertices. In the current article we are going to confine ourselves to the case of essentially bounded potentials and standard boundary conditions. The case of L_1 potentials and most general symmetric boundary conditions will be considered in the following publication, since these problems can be treated by similar methods.

To obtain a connection between the spectrum of a Schrödinger operator and the Euler characteristic of the graph we use trace formula. A similar formula was first proven by J.-P. Roth [33] using the heat kernel expansion, but we are going to use the trace formula in the form (2) first presented (without a proof) by J.-P. Roth as well. The formula we are going to use was first given by T. Kottos and U. Smilansky, but without paying attention to the fact that the secular equation describing the spectrum using vertex and edge scattering matrices in general does not determine the correct multiplicity of the eigenvalue zero. Correction of this inaccuracy allowed us to prove that two isospectral graphs (the corresponding Laplacians have the same spectra) have the same Euler characteristic and provide an effective formula for it. In the current article we not only prove a slightly different formula for the Euler characteristic, but give an alternative proof of this formula without any use of the trace formula, but only for graphs with the edges being integer multiples of one length to be called the basic length. We hope that this approach provides a new insight in the spectral asymptotics for such operators.

The new formula for the Euler characteristic obtained in this paper is valid not only for Laplacians but for Schrödinger operators with essentially bounded potentials and standard boundary conditions. To prove this fact we show first that the Euler characteristics is determined by the asymptotics of the eigenvalues only. Then it remains to prove that the spectra of a Laplacian and the corresponding Schrödinger operator are asymptotically close and therefore the formula for the Euler characteristic originally proven for Laplacians gives the correct result if the spectrum of the Laplacian is just substituted with the spectrum of the Schrödinger operator.

Current paper extends the recent article [26] and therefore correct references and historical remarks can be found there. On the other hand, it is impossible not to mention that the theory of differential operators on graphs has been grown from the pioneering works by B. Pavlov, N. Gerasimenko [15,16], P. Exner, P. Šeba [13] and Y. Colin de Verdière [11,12]. Recent interest in this subject was initiated by papers by V. Kostrykin and R. Schrader [19–21]. Several articles have been devoted to differential operators on trees—special class of graphs without cycles. It is possible to state that this problem is fully understood now [1,3,4,9,27,34,36,37]. On the other hand, very few papers are devoted to operators on arbitrary graphs (with cycles) and here we would like to mention J. von Below [5], R. Carlson [10], L. Friedlander [14], B. Gutkin and U. Smilansky [18], V. Kostrykin and R. Schrader [22], J. Boman, P. Kurasov, F. Stenberg and M. Nowaczyk [6,24–26,30,31]. The results presented in this article can be considered as a natural generalization of the classical results on inverse spectral and scattering problems for