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Noncommutative Berezin transforms and multivariable operator model theory ☆

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Abstract

In this paper, we initiate the study of a class $\mathbf{D}_p^m(\mathcal{H})$ of noncommutative domains of n-tuples of bounded linear operators on a Hilbert space $\mathcal{H}$, where $m \geqslant 2$, $n \geqslant 2$, and p is a positive regular polynomial in n noncommutative indeterminates. These domains are defined by certain positivity conditions on p, i.e.,

$$\mathbf{D}_p^m(\mathcal{H}) := \{X := (X_1, \dots, X_n):\ (1-p)^k(X, X^*) \geqslant 0 \text{ for } 1 \leqslant k \leqslant m\}.$$

Each such a domain has a universal model $(W_1, \dots, W_n)$ of weighted shifts acting on the full Fock space $F^2(H_n)$ with n generators. The study of $\mathbf{D}_p^m(\mathcal{H})$ is close related to the study of the weighted shifts $W_1, \dots, W_n$, their joint invariant subspaces, and the representations of the algebras they generate: the domain algebra $\mathcal{A}_n(\mathbf{D}_p^m)$, the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$, and the C^*-algebra $C^*(W_1, \dots, W_n)$. A good part of this paper deals with these issues.

The main tool, which we introduce here, is a noncommutative Berezin type transform associated with each n-tuple of operators in $\mathbf{D}_p^m(\mathcal{H})$. The study of this transform and its boundary behavior leads to Fatou type results, functional calculi, and a model theory for n-tuples of operators in $\mathbf{D}_p^m(\mathcal{H})$. These results extend to noncommutative varieties $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H}) \subset \mathbf{D}_p^m(\mathcal{H})$ generated by classes $\mathcal{Q}$ of noncommutative polynomials. When $m \geqslant 2$, $n \geqslant 2$, $p = Z_1 + \cdots + Z_n$, and $\mathcal{Q} = 0$, the elements of the corresponding variety $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions.

Our results apply, in particular, when $\mathcal{Q}$ consists of the noncommutative polynomials $Z_iZ_j - Z_jZ_i$, $i, j = 1, \dots, n$. In this case, the model space is a symmetric weighted Fock space $F_s^2(\mathbf{D}_p^m)$, which is identified with a reproducing kernel Hilbert space of holomorphic functions on a Reinhardt domain in $\mathbb{C}^n$, and the universal model is the n-tuple $(M_{\lambda_1}, \dots, M_{\lambda_n})$ of multipliers by the coordinate functions. In this particular

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case, we obtain a model theory for commuting n-tuples of operators in $\mathbf{D}_p^m(\mathcal{H})$, recovering several results already existent in the literature.

Keywords: Multivariable operator theory; Noncommutative domain; Noncommutative variety; Dilation theory; Model theory; Weighted shift; Wold decomposition; Fock space; von Neumann inequality; Berezin transform; Creation operators

0. Introduction

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0, and consider a polynomial $q = q(Z_1, \ldots, Z_n) = \sum c_\alpha Z_\alpha$ in noncommutative indeterminates $Z_1, \ldots, Z_n$, where we denote $Z_\alpha := Z_{i_1} \ldots Z_{i_k}$ if $\alpha = g_{i_1} \ldots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and $Z_{g_0} := I$. We associate with q the operator

$$q(X, X^*) := \sum c_\alpha X_\alpha X_\alpha^*,$$

where $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ and $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Let $p = p(Z_1, \ldots, Z_n) = \sum a_\alpha Z_\alpha$, $a_\alpha \in \mathbb{C}$, be a positive regular polynomial, i.e., $a_\alpha \geqslant 0$, $a_{g_0} = 0$, and $a_{g_i} > 0$, $i = 1, \ldots, n$. Given $m, n \in \{1, 2, \ldots\}$, we define the noncommutative domain

$$\mathbf{D}_p^m(\mathcal{H}) := \bigl\{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon\ (1-p)^k(X, X^*) \geqslant 0 \text{ for } 1 \leqslant k \leqslant m\bigr\}.$$

In the last fifty years, these domains have been studied in several particular cases. Most of all, we should mention that the study of the closed operator unit ball

$$\bigl[B(\mathcal{H})\bigr]_1^- := \bigl\{X \in B(\mathcal{H})\colon\ I - XX^* \geqslant 0\bigr\}$$

(which corresponds to the case $m = 1$, $n = 1$, and $p = Z$) has generated the celebrated Sz.-Nagy–Foias theory of contractions on Hilbert spaces and has had profound implications in function theory, interpolation, prediction theory, scattering theory, and linear system theory (see [11,21,22,51], etc.). The case when $m = 1$, $n \geqslant 2$, and $p = Z_1 + \cdots + Z_n$, corresponds to the closed operator ball

$$\bigl[B(\mathcal{H})^n\bigr]_1^- := \bigl\{(X_1, \ldots, X_n) \in B(\mathcal{H})^n\colon\ I - X_1X_1^* - \cdots - X_nX_n^* \geqslant 0\bigr\}$$

and its study has generated a *free* analogue of Sz.-Nagy–Foias theory (see [10,13,19,23,30–43,46,47], etc.). The commutative case, which corresponds to the subvariety of $[B(\mathcal{H})^n]_1$ determined by the commutators $Z_iZ_j - Z_jZ_i$, $i, j = 1, \ldots, n$, was considered by Drurry [20], extensively studied by Arveson [7,8], and considered by the author [39] in connection with noncommutative Poisson transforms. More general subvarieties in $[B(\mathcal{H})^n]_1$, determined by classes of noncommutative polynomials, were considered by the author in [43,46]. The study of the unit ball $[B(\mathcal{H})^n]_1$ was extended, in [45], to noncommutative domains $\mathbf{D}_p^m(\mathcal{H})$ (respectively subvarieties) when $m = 1$, $n \geqslant 1$, and p is any positive regular noncommutative polynomial (respectively free holomorphic function in the sense of [44]).