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Radial solutions for the Brezis–Nirenberg problem involving large nonlinearities ☆

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Abstract

Let us consider the problem

$$\begin{cases} -\Delta u + a\big(|x|\big)u = u^p & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{0.1}$$

where B_1 is the unit ball in $\mathbb{R}^N$, $N \geqslant 3$, and $a(|x|) \geqslant 0$ is a smooth radial function.

Under some suitable assumptions on the regular part of the Green function of the operator $-u'' - \frac{N-1}{r}u + a(r)u$, we prove the existence of a radial solution to (0.1) for p large enough.

Keywords: Supercritical problems; Green's function; Radial solutions

1. Introduction

Let us consider the problem

$$\begin{cases} -\Delta u + a(x)u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 3$, $p > 1$, $a(x)$ is a smooth function and the operator $-\Delta + a(x)I$ is coercive.

Let us recall some existence results to (1.1). First we consider the subcritical case, i.e. $1 < p < \frac{N+2}{N-2}$. In this setting there always exists a solution to (1.1) which can be found as

$$\inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{p+1}=1}} \int_\Omega \left(|\nabla u|^2 + a(x)u^2\right).$$

We point out that the compactness of embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ for $1 < p < \frac{N+2}{N-2}$ plays a crucial role.

If $p = \frac{N+2}{N-2}$ (the critical case) the problem becomes much more difficult. Indeed, using the Pohozaev identity [22] it is possible to show that if Ω is star-shaped with respect to some point and $a(x) \equiv 0$ then there is no solution to (1.1). So it is not possible to obtain the same result as in the subcritical case.

A relevant progress in investigating the critical case was done in the pioneering paper of Brezis and Nirenberg [3]. Among the other results they proved that for $N \geqslant 4$ and $a(x) \leqslant \delta < 0$ on some open subset of Ω there exists at least one solution to (1.1).

A different sufficient condition to ensure solutions to (1.1) for positive $a(x)$ can be found in [20] (see also [11] for the case $N = 3$).

Another fundamental result in the critical case $p = \frac{N+2}{N-2}$ and $a(x) \equiv 0$ is due to Coron [4], Bahri and Coron [1] where is showed the role of topology of the domain in the existence of solutions to (1.1). In particular, if Ω has one hole, there exists a solution to (1.1).

The supercritical case $p > \frac{N+2}{N-2}$ appears more complicated and there is no existence result for general domain (or suitable function $a(x)$) for any $p > 1$. However, let us recall that

(i) if Ω is star-shaped with respect to some point and $a(x) = \lambda \geqslant 0$ then there is no solution to (1.1);
(ii) if Ω is an annulus and $a(x) \equiv 0$ Kazdan and Warner [16] proved the existence of a radial solution for any $p > 1$.

On the other hand, in the last years there was some progress considering $p = \frac{N+2}{N-2} + \varepsilon$ where ε is a (small) positive parameter. We just mention the papers [2,6–8,14,17,21] and the references therein.

For other results which are not a perturbation of the critical case, we mention some interesting existence and nonexistence results due to Passaseo [18,19]. In these papers was constructed a contractible domain for which there is a solution to (1.1) for any $p > 1$ and it was exhibited a nontrivially topological domain for which there is no solution to (1.1). This shows that the Bahri–Coron result cannot be true in the supercritical case.

We also quote a recent result due to del Pino and Wei [5] where the authors prove that if Ω is a domain with a small hole and $a(x) \equiv 0$ then for any $p > \frac{N+2}{N-2}$, $p \neq p_n$, where p_n is a suitable sequence such that $p_n \to \infty$ as $n \to \infty$, there exists at least one solution.