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Oscillatory integral operators with homogeneous polynomial phases in several variables

Allan Greenleaf [a,∗,1], Malabika Pramanik [b,1], Wan Tang [c]

[a] *Department of Mathematics, University of Rochester, Rochester, NY 14627, USA*
[b] *Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2*
[c] *Department of Biostatistics, University of Rochester, Rochester, NY 14642, USA*

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Abstract

We obtain L^2 decay estimates in λ for oscillatory integral operators T_λ whose phase functions are homogeneous polynomials of degree m and satisfy various genericity assumptions. The decay rates obtained are optimal in the case of $(2+2)$-dimensions for any m, while in higher dimensions the result is sharp for m sufficiently large. The proof for large m follows from essentially algebraic considerations. For cubics in $(2+2)$-dimensions, the proof involves decomposing the operator near the conic zero variety of the determinant of the Hessian of the phase function, using an elaboration of the general approach of Phong and Stein [D.H. Phong, E.M. Stein, Models of degenerate Fourier integral operators and Radon transforms, Ann. of Math. (2) 140 (1994) 703–722].

Keywords: Oscillatory integral operators; Decay estimates; Polynomial phase function; Newton polyhedron

∗ Corresponding author. Fax: +1 585 273 4655.
E-mail addresses: allan@math.rochester.edu (A. Greenleaf), malabika@math.ubc.ca (M. Pramanik), wtang@bst.rochester.edu (W. Tang).
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1. Introduction

Consider an oscillatory integral operator

$$T_\lambda f(x) = \int_{\mathbb{R}^{n_Z}} e^{i\lambda S(x,z)} a(x,z) f(z)\, dz, \quad x \in \mathbb{R}^{n_X}, \tag{1.1}$$

where S is a real-valued phase function on $\mathbb{R}^{n_X} \times \mathbb{R}^{n_Z}$, $a \in C_0^\infty(\mathbb{R}^{n_X} \times \mathbb{R}^{n_Z})$ is a fixed amplitude supported in a compact neighborhood of the origin, and λ is a large parameter. For λ fixed, T_λ defines a bounded operator from $L^2(\mathbb{R}^{n_Z})$ to $L^2(\mathbb{R}^{n_X})$. We refer to this setting as "$(n_X + n_Z)$-dimensions." A basic problem arising in many contexts [5,14,15] is determining the optimal rate of decay of the L^2 operator norm $\|T_\lambda\|$ as $\lambda \to \infty$. Typically, an upper bound for $\|T_\lambda\|$ is of the form

$$\|T_\lambda\| \leqslant C\lambda^{-r} (\log \lambda)^p, \quad \lambda \to \infty,$$

with $r > 0$ and $p \geqslant 0$ depending on S. For $n_X = n_Z = 1$, sharp results were obtained for C^ω phases by Phong and Stein [11], with the decay rate determined by the Newton polygon of $S(x,z)$. This was extended to most C^∞ phases by Rychkov [12], with the remaining cases settled by Greenblatt [3]. See also Seeger [13].

Extending all of these results to higher dimensions seems a difficult undertaking, and in the current work we focus on a more approachable problem, namely finding higher-dimensional analogues of the results in Phong and Stein [9,10] concerning homogeneous polynomials in $(1+1)$-dimensions. One can assume that the phase function does not contain any monomial terms that are purely functions of x or of z, since these do not affect the L^2 operator norm, and then the main result of [10] is:

Theorem A *(Phong and Stein). Let $n_X = n_Z = 1$ and $S(x,z) = \sum_{j=1}^{m-1} a_j x^j z^{m-j}$. Assume that there exist $j \leqslant m/2$ and $k \geqslant m/2$ such that $a_j \neq 0$ and $a_k \neq 0$. Then*

$$\|T_\lambda\| \leqslant C\lambda^{-1/m}, \quad \lambda \to +\infty.$$

This result has been partially extended to $(2+1)$-dimensions by Tang [18]. (See also Fu [1], where certain homogeneous polynomial phases, linear in one of the variables, are considered.) The setup in [18] is as follows: write

$$S(x,z) = \sum_{j=1}^{m-1} P_j(x_1, x_2) z^{m-j},$$

where the P_j are homogeneous forms of degree j on $\mathbb{R}^2$. Recall that a form P is nondegenerate if $\nabla P(x) \neq 0$ for $x \neq 0$; this is equivalent with P factoring over $\mathbb{C}$ into $\deg(P)$ distinct linear factors. Let $j_{\min}$ (respectively, $j_{\max}$) denote the first (respectively, last) index j for which P_j is not identically zero. The main result of [18] is: