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ScienceDirect
Journal of Functional Analysis 254 (2008) 396–440
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On tracial approximation

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Received 26 February 2007; accepted 7 August 2007

Available online 3 December 2007

Communicated by P. Delorme

Abstract

Let $\mathscr{C}$ be a class of unital C*-algebras. The class TA$\mathscr{C}$ of C*-algebras which can be tracially approximated (in the Egorov-like sense first considered by Lin) by the C*-algebras in $\mathscr{C}$ is studied (Lin considered the case that $\mathscr{C}$ consists of finite-dimensional C*-algebras or the tensor products of such with C([0, 1])). In particular, the question is considered whether, for any simple separable $A \in \mathrm{TA}\mathscr{C}$, there is a C*-algebra B which is a simple inductive limit of certain basic homogeneous C*-algebras together with C*-algebras in $\mathscr{C}$, such that the Elliott invariant of A is isomorphic to the Elliott invariant of B. An interesting case of this question is answered. In the final part of the paper, the question is also considered which properties of C*-algebras are inherited by tracial approximation. (Results of this kind are obtained which are used in the proof of the main theorem of the paper, and also in the proof of the classification theorem of the second author given in [Z. Niu, A classification of tracially approximately splitting tree algebra, in preparation] and [Z. Niu, A classification of certain tracially approximately subhomogeneous C*-algebras, PhD thesis, University of Toronto, 2005]—which also uses the main result of the present paper.)

Keywords: Elliott invariant; Tracial approximation

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[1] The research of the author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.08.005

1. Introduction

The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF-algebras [8]. (This was closely related to the Bratteli diagram classification given earlier in [3], but differed in introducing the K-functor.) Since then (but only after a fifteen-year hiatus), many classes of amenable C*-algebras have been found to be classified by their Elliott invariants. Among them, one important class is the class of simple unital inductive limits of homogeneous C*-algebras (AH-algebras for short). In this paper, by a unital homogeneous C*-algebra, we refer to a C*-algebra which is isomorphic to

$$p\mathrm{M}_n\big(\mathrm{C}(X)\big)p$$

for some compact Hausdorff space X, and some projection p in $\mathrm{M}_n(\mathrm{C}(X))$. (Noted that these C*-algebras are exactly the homogeneous C*-algebras with trivial Dixmier–Douady class. See, for example, p. 14 of [1]. In general, a homogeneous C*-algebra may not have this form.) In [16] and [14], Elliott, Gong, and Li showed that C*-algebras in this class can be classified by their Elliott invariant, provided that the dimensions of the base spaces of their building blocks are uniformly bounded. (Such AH-algebras are referred as AH-algebras without dimension growth.) Many naturally arising C*-algebras—for instance, the irrational rotation C*-algebras [12]—are known to be AH-algebras without dimension growth. (Note that AH-algebras not in this class were constructed by Villadsen in [33] and [34].)

A very important axiomatic version of the classification of the AH-algebras without dimension growth was given by Huaxin Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth [18, 20,21,23]. More precisely, Lin introduced the class of tracially approximate interval algebras—TAI-algebras for short. Recall that an interval algebra is a C*-algebra isomorphic to $F\otimes\mathrm{C}([0,1])$ for a finite-dimensional C*-algebra F. Then TAI-algebras are defined by the following.

Definition. A unital C*-algebra A is a TAI-algebra if for any $\varepsilon>0$, any finite subset $\mathcal{F}\subseteq A$, and any non-zero $a\in A^+$, there exist a non-zero projection $p\in A$ and a sub-C*-algebra $I\subseteq A$ such that I is an interval algebra, $1_I=p$, and for all $x\in\mathcal{F}$,

(1) $\|xp-px\|<\varepsilon$,
(2) there exists $x'\in I$ such that $\|pxp-x'\|<\varepsilon$, and
(3) $1-p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The classification theorem for TAI-algebras was given by Lin in [23]. (The second author also contributed to the final part of this work; see [26].)

Theorem. *(See [23].) Let A and B be two simple amenable unital TAI-algebras satisfying the UCT. Then A is isomorphic to B if the Elliott invariant of A is isomorphic to the Elliott invariant of B. Moreover, the isomorphism between the algebras can be chosen in such a way that it induces the given isomorphism between the invariants.*

Since AH-algebras without dimension growth are TAI-algebras by the work of Gong [16], and since AH-algebras exhaust the invariant for the class in question (see [32], combined with