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Bivariant Chern character and longitudinal index

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Abstract

In this paper we consider a family of Dirac-type operators on fibration $P \to B$ equivariant with respect to an action of an étale groupoid. Such a family defines an element in the bivariant K theory. We compute the action of the bivariant Chern character of this element on the image of Connes' map Φ in the cyclic cohomology. A particular case of this result is Connes' index theorem for étale groupoids [A. Connes, Noncommutative Geometry, Academic Press, 1994] in the case of fibrations.

Keywords: Cyclic cohomology; Index theory; Chern character

1. Introduction

In [36,37], answering a question posed by A. Connes in [6], V. Nistor defined a bivariant Chern character for a p-summable quasihomomorphism and established its fundamental properties. This theory was further developed in [16,17]. Bivariant Chern character encodes extensive information related to index theory. In the present paper we compute the action of the bivariant Chern character on cyclic cohomology in geometric situations arising from equivariant families of elliptic operators.

We consider first the following geometric situation. Let $\pi : P \to B$ be a fibration. We assume that we are given a vertical Riemannian metric on this fibration, i.e. that there is a Riemannian metric on each of the fibers $P_b = \pi^{-1}(b)$, which varies smoothly with b. We assume that each P_b is a complete Riemannian manifold. Let D be a family of fiberwise Dirac-type operators on

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this fibration acting on the section of the bundle $\mathcal{E}$. In this paper we consider only the even-dimensional situation, and hence the bundle $\mathcal{E}$ is always $\mathbb{Z}_2$-graded. We assume that an étale groupoid G with the unit space B acts on this fibration, and that the operator D is invariant under this action. This means that every $\gamma \in G$ defines a diffeomorphism $P_{r(\gamma)} \to P_{s(\gamma)}$, $p \mapsto p\gamma$. One also requires that these diffeomorphisms are compatible with the groupoid structure, i.e. that $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$. Notice that this implies that the vertical metric is invariant under the action of G. We do not assume existence of a metric on P, invariant under the action of G.

There are two constructions in K-theory and cyclic cohomology we need to use to describe the problem. The first is Nistor's bivariant Chern character. It appears in our setting as follows. The operator D described above defines an element of the equivariant KK-theory $KK^G(C_0(P), C_0(B))$ [29,30]. Using the canonical map

$$j^G : KK^G\big(C_0(P), C_0(B)\big) \to KK\big(C_0(P) \rtimes G, C_0(B) \rtimes G\big) \tag{1}$$

we obtain a class in $KK(C_0(P) \rtimes G, C_0(B) \rtimes G)$ defined by D. This class can be represented by an explicit quasihomomorphism ψ_D in the sense of J. Cuntz [14,15]. Moreover, one can show that ψ_D actually defines a *p-summable* quasihomomorphism of smooth algebraic cross-products $C_0^\infty(P) \rtimes G$ and $C_0^\infty(B) \rtimes G$ in the sense of V. Nistor [36,37]. One can then use techniques developed in [36,37] to define the bivariant Chern character $\mathrm{Ch}(D)$ in bivariant cyclic homology which is a morphism of complexes $CC_*(C_0^\infty(P) \rtimes G) \to CC_*(C_0^\infty(B) \rtimes G)$.

The other tool which we need is the explicit construction, due to A. Connes [8,9], of the classes in the cyclic cohomology of cross product algebras. Let M be a manifold on which an étale groupoid G acts, and let M_G be the corresponding homotopy quotient. Then Connes constructs an explicit chain map of complexes inducing an injective map in cohomology $\Phi : H_\tau^*(M_G) \to HC^*(C_0^\infty(M) \rtimes G)$. Here τ denotes a twisting by the orientation bundle of M. Introduce now the following notations. The map π induces a map $P_G \to B_G = BG$ which we also denote by π. Since the fibers of $P_G \to B_G$ are *spin*^{c} and hence oriented we have a pull-back map $\pi^* : H_\tau^*(B_G) \to H_\tau^*(P_G)$. Let $\widehat{A}_G(TP/B) \in H^*(P_G)$ be the equivariant $\widehat{A}$-genus of the vertical tangent bundle, i.e. the $\widehat{A}$-genus of the bundle on P_G induced by the vertical tangent bundle TP/B on P. Note that we use the conventions from [1] in the definitions of characteristic classes. Let $\mathrm{Ch}_G(\mathcal{E}/\mathcal{S})$ be the equivariant twisting Chern character of the bundle $\mathcal{E}$. If the fibers of $P \to B$ have *spin* structure, and $\mathcal{E} = \mathcal{S} \otimes V$ where $\mathcal{S}$ is the vertical spin bundle, then $\mathrm{Ch}_G(\mathcal{E}/\mathcal{S}) = \mathrm{Ch}_G(V)$ is just the equivariant Chern character of V. In other words it is the Chern character of the bundle on P_G induced by the bundle V on P. Set now for $c \in H_\tau^*(B_G)$

$$\tilde{\pi}^*(c) = (2\pi i)^{-\frac{\dim P - \dim B}{2}} \widehat{A}_G(TP/B)\,\mathrm{Ch}_G(\mathcal{E}/\mathcal{S})\pi^*(c).$$

Our main result is then the following theorem.

Theorem 1. *The diagram*

$$\begin{array}{ccc} H_\tau^*(P_G) & \xrightarrow{\ \Phi\ } & HC^*(C_0^\infty(P) \rtimes G) \\ \tilde{\pi}^* \Big\uparrow & & \Big\uparrow \mathrm{Ch}(D)^t \\ H_\tau^*(B_G) & \xrightarrow{\ \Phi\ } & HC^*(C_0^\infty(B) \rtimes G) \end{array} \tag{2}$$

commutes.