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Berezin symbol and invertibility of operators on the functional Hilbert spaces

Mubariz T. Karaev

Suleyman Demirel University, Isparta (MYO) Vocation School, 32260 Isparta, Turkey

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Abstract

We give in terms of reproducing kernel and Berezin symbol the sufficient conditions ensuring the invertibility of some linear bounded operators on some functional Hilbert spaces.

Keywords: Hardy space; Bergman space; Toeplitz operator; Berezin symbol; Reproducing kernel

1. Introduction

Let $\mathbf{T}$ be the unit circle $\mathbf{T} = \{\boldsymbol{\zeta} \in \mathbf{C}\colon\ |\boldsymbol{\zeta}| = 1\}$, $\varphi \in L^\infty = L^\infty(\mathbf{T})$, and let T_φ be the Toeplitz operator acting in the Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbf{C}\colon\ |z| < 1\}$ by the formula $T_\varphi f = P_+\varphi f$, where P_+ is the Riesz projector. Let $\tilde{\varphi}$ denote the harmonic extension of the function φ to $\mathbb{D}$. In [5] Douglas posed the following problem: if φ is a function in L^∞ for which $|\tilde{\varphi}(z)| \geqslant \delta > 0$, $z \in \mathbb{D}$, then is T_φ invertible?

In [18] Tolokonnikov firstly gave a positive answer to this question under the condition that δ is near enough to 1, namely, he proved that if

$$1 \geqslant \left|\tilde{\varphi}(z)\right| \geqslant \delta > \frac{45}{46}, \quad z \in \mathbb{D},$$

then T_φ is invertible and

$$\left\|T_\varphi^{-1}\right\| \leqslant \left(1 - 46(1-\delta)\right)^{-1}.$$

E-mail address: garayev@fef.sdu.edu.tr.

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This assertion was also proved by Wolff [19]. Nikolskii [15] has somewhat improved the result of Tolokonnikov proving invertibility of T_φ and the estimate

$$\left\| T_\varphi^{-1} \right\| \leqslant (24\delta - 23)^{-1/2}$$

under condition $\delta > 23/24$. Finally, Wolff [19] has constructed a function $\varphi \in L^\infty$ such that $\inf_{\mathbf{D}} |\tilde{\varphi}(z)| > 0$ but the corresponding operator T_φ is not invertible, and thus showed that the answer to the question of Douglas is negative in general. Since $\tilde{\varphi}$ coincides with the Berezin symbol $\widetilde{T}_\varphi$ of the operator T_φ (see Lemma 2.1), in this context the following natural problem arises.

Problem 1. *Let A be a linear bounded operator acting in the functional Hilbert space $\mathcal{H}(\Omega)$ of complex-valued functions over the some (non-empty) set Ω, such that $|\widetilde{A}(z)| \geqslant \delta$ for all $z \in \Omega$ and for some $\delta > 0$. To find the number δ_0, which can be (more or less) easily computed from the data of A, and due to which the inequality*

$$\left|\widetilde{A}(z)\right| \geqslant \delta > \delta_0, \quad z \in \mathbf{D},$$

ensures the invertibility of A, where $\widetilde{A}$ denotes the Berezin symbol of the operator A.

In particular, the following problem is also interesting, which is closely related with the finite section method of Böttcher and Silbermann [3].

Problem 2. *Let $E \subset \mathcal{H}(\Omega)$ be a closed subspace of the functional Hilbert space $\mathcal{H}(\Omega)$, and let A be a linear bounded operator acting in $\mathcal{H}(\Omega)$ such that*

$$\left|\widetilde{A}(z)\right| \geqslant \delta$$

for all $z \in \Omega$ and for some $\delta > 0$. To find a number δ_0, such that $\delta > \delta_0$ ensures the invertibility of operator $P_E A \mid E$ (the compression of the operator A to the subspace E), where P_E is an orthogonal projection from $\mathcal{H}(\Omega)$ onto E.

In this article we solve these problems in some special cases. Our argument uses the concept of reproducing kernel and Berezin symbol.

2. Notations and preliminaries

2.1. Recall that a functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (non-empty) set Ω, which has the property that point evaluations are continuous (i.e., for each $\lambda \in \Omega$, the map $f \to f(\lambda)$ is a continuous linear functional on $\mathcal{H}$). Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda\colon \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. It is well known (see, for instance, [8, Problem 37] that if $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z).$$