Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Functional Analysis 238 (2006) 353–373
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On the number of permutatively inequivalent basic sequences in a Banach space

Valentin Ferenczi *

Equipe d'Analyse Fonctionnelle, Université Pierre et Marie Curie – Paris 6, Boîte 186, 4, Place Jussieu, 75252, Paris cedex 05, France

Received 3 October 2005; accepted 23 January 2006

Available online 24 February 2006

Communicated by G. Pisier

Abstract

Let X be a Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$. The relation E_0 is Borel reducible to permutative equivalence between normalized block-sequences of $(e_n)_{n\in\mathbb{N}}$ or X is c_0 or ℓ_p saturated for some $1 \leqslant p < +\infty$. If $(e_n)_{n\in\mathbb{N}}$ is shrinking unconditional then either it is equivalent to the canonical basis of c_0 or ℓ_p, $1 < p < +\infty$, or the relation E_0 is Borel reducible to permutative equivalence between sequences of normalized disjoint blocks of X or of X^*. If $(e_n)_{n\in\mathbb{N}}$ is unconditional, then either X is isomorphic to ℓ_2, or X contains 2^ω subspaces or 2^ω quotients which are spanned by pairwise permutatively inequivalent normalized unconditional bases.

Keywords: Borel reducibility; Permutative equivalence; Block basis; Homogeneous space; Dichotomy

1. Introduction

In the 1990s, W.T. Gowers and R. Komorowski–N. Tomczak-Jaegermann solved the so-called Homogeneous Banach Space Problem. A Banach space is said to be homogeneous if it is isomorphic to its infinite-dimensional closed subspaces; it is a consequence of two theorems proved by these authors that a homogeneous Banach space must be isomorphic to ℓ_2 [14,21].

It is then natural to ask how many non-isomorphic subspaces a given Banach space must contain when it is not isomorphic to ℓ_2. This question was first asked the author by G. Godefroy,

* Fax: +33 1 44 27 25 55.
E-mail address: ferenczi@ccr.jussieu.fr.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.01.011

and not much was known until recently about it in the literature, even concerning the classical spaces c_0 and ℓ_p.

The correct setting for this question is the classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [13] and independently from the works of L.A. Harrington, A.S. Kechris and A. Louveau [18]. It may be thought of as an extension of the notion of cardinality in terms of complexity, when one counts equivalence classes.

A topological space is Polish if it is separable and its topology may be generated by a complete metric. Its Borel subsets are those belonging to the smallest σ-algebra containing the open sets. An analytic subset is the continuous image of a Polish space, or equivalently, of a Borel subset of a Polish space.

If R (respectively S) is an equivalence relation on a Polish space E (respectively F), then it is said that (E, R) is Borel reducible to (F, S) if there exists a Borel map $f : E \to F$ such that $\forall x, y \in E, xRy \Leftrightarrow f(x)Sf(y)$. An important equivalence relation is the relation E_0: it is defined on 2^ω by

$$\alpha E_0 \beta \quad \Leftrightarrow \quad \exists m \in \mathbb{N},\ \forall n \geqslant m, \quad \alpha(n) = \beta(n).$$

The relation E_0 is a Borel equivalence relation with 2^ω classes and which, furthermore, admits no Borel classification by real numbers, that is, there is no Borel map f from 2^ω into $\mathbb{R}$ (equivalently, into a Polish space), such that $\alpha E_0 \beta \Leftrightarrow f(\alpha) = f(\beta)$; such a relation is said to be *non-smooth*. In fact E_0 is the $\leqslant_B$ minimum non-smooth Borel equivalence relation [18].

There is a natural way to equip the set of subspaces of a Banach space X with a Borel structure (see, e.g., [20]), and the relation of isomorphism is analytic in this setting [2]. The relation E_0 then appears as a natural threshold for results about isomorphism between separable Banach spaces. A Banach space X was defined in [11] to be *ergodic* if E_0 is Borel reducible to isomorphism between subspaces of X; in particular, an ergodic Banach space has continuum many non-isomorphic subspaces, and isomorphism between its subspaces is non-smooth.

The question of the complexity of isomorphism between subspaces of a given Banach space X is related to results and questions of Gowers about the structure of the relation of embedding between subspaces of X [14]. In that article, Gowers proves the following structure theorem.

Theorem 1.1 *(W.T. Gowers). Any Banach space contains a subspace Y satisfying one of the following properties, which are mutually exclusive and all possible*:

(a) *Y is hereditarily indecomposable* (*i.e. contains no direct sum of infinite-dimensional subspaces*);
(b) *Y has an unconditional basis and no disjointly supported subspaces of Y are isomorphic*;
(c) *Y has an unconditional basis and is strictly quasi-minimal* (*i.e. any two subspaces of Y have further isomorphic subspaces, but Y contains no minimal subspace*);
(d) *Y has an unconditional basis and is minimal* (*i.e. Y embeds into any of its subspaces*).

Note that these properties are preserved by passing to block-subspaces (in the associated natural basis). Furthermore, knowing that a space belongs to one of the classes (a)–(d) gives a lot of informations about operators and isomorphisms defined on it (see [14] about this).