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Journal of Functional Analysis 237 (2006) 565–588
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On radial stochastic Loewner evolution in multiply connected domains

Robert O. Bauer [a,*,1], Roland M. Friedrich [b,2]

[a] *Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA*

[b] *Institute for Advanced Study, Princeton, NJ 08540, USA*

Received 8 August 2005; accepted 29 December 2005

Available online 3 February 2006

Communicated by Paul Malliavin

Abstract

We discuss the extension of radial SLE to multiply connected planar domains. First, we extend Loewner's theory of slit mappings to multiply connected domains by establishing the radial Komatu–Loewner equation, and show that a simple curve from the boundary to the bulk is encoded by a motion on moduli space and a motion on the boundary of the domain. Then, we show that the vector-field describing the motion of the moduli is Lipschitz. We explain why this implies that "consistent," conformally invariant random simple curves are described by multidimensional diffusions, where one component is a motion on the boundary, and the other component is a motion on moduli space. We argue what the exact form of this diffusion is (up to a single real parameter κ) in order to model boundaries of percolation clusters. Finally, we show that this moduli diffusion leads to random non-self-crossing curves satisfying the locality property if and only if $\kappa = 6$.

Keywords: Radial SLE; Multiply connected domains; Moduli diffusion

* Corresponding author.
E-mail addresses: rbauer@math.uiuc.edu, rbauer13@uiuc.edu (R.O. Bauer), rolandf@ias.edu (R.M. Friedrich).

1 The research was supported by NSA grant H98230-04-1-0039.
2 The research was supported by NSF grant DMS-0111298.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2005.12.023

1. Introduction

In this paper we discuss conformally invariant growing random compact sets in multiply connected domains. Our results are meant to provide some of the steps to extend the stochastic Loewner evolution of Schramm [10], from simply connected domains to multiply connected domains and Riemann surfaces. Based on Loewner's theory of slit mappings, Schramm showed that conformally invariant measures describing random "simple" curves in simply connected domains, can be encoded into a diffusion on the boundary of the domain if the random simple curves satisfy a "consistency condition." Furthermore, under an additional but very natural symmetry condition, he showed that the diffusion on the boundary is a multiple of Brownian motion if the random simple curve lives in certain standard domains.

We show that consistent and conformally invariant random simple curves in multiply connected domains can be encoded into a multidimensional diffusion. One component of this diffusion corresponds to a motion on the boundary of the domain and the other components are the moduli of the domain with the random simple curve, grown up to time t, removed. The random simple curves we consider in this paper connect the boundary to the bulk (interior) and so the appropriate moduli space is the moduli space of n-connected domains with one marked interior point and one marked boundary point. We show that consistency (a Markovian-type property) and conformal invariance essentially determine the diffusion up to the drift of the motion on the boundary. We call such diffusions on moduli space Schiffer diffusions. Under a symmetry condition, familiar from percolation, this drift component can also be identified, leaving a single real parameter κ. Beginning with the Schiffer diffusion with this drift we show that the resulting family of random growing compact sets satisfies the locality property if $\kappa = 6$.

The fundamental observation that diffusion processes on the moduli space of bordered Riemann surfaces with marked points, given its path-wise solutions agree with the geometric constraints, yield the most general way to define probability measures on (simple) curves on surfaces and therefore contain "ordinary" SLE as a special case, was introduced in [3,6]. However, the current article is the first constructive implementation of it, for the radial case and multiply connected domains. For general Riemann surfaces and the chordal case one proceeds along similar lines as described here, but with some necessary modifications. An important role then is played by the so-called "Hilbert uniformisation."

The paper is structured as follows. In Section 3 we introduce a suitable family of standard domains and describe canonical mappings onto these domains in terms of the Green function and associated functions. In Section 4 we introduce an appropriate time parameter for the Jordan arcs in a multiply connected domain, namely the conformal radius, and establish a variational formula for "increments" of the conformal radius under perturbations of the domain. In Section 5 we establish what we call the radial Komatu–Loewner equation, which generalizes the radial Loewner equation to multiply connected domains. In Section 6 we obtain the corresponding equation for the evolution of the moduli and then show that the vector field in the differential equation satisfies a Lipschitz property, Theorem 2. This result is crucial as it allows us to reverse the construction: start with a motion on the boundary, solve the equation for the moduli for the given boundary motion, then solve the radial Komatu–Loewner equation to obtain a growing family of compacts. In Section 7 we use the correspondence between growing "simple" curves and paths on moduli space to show that consistent conformally invariant random simple curves are given by diffusions on moduli space, and identify the diffusion up to a single drift term. Under an additional symmetry condition we then identify the diffusion up to a single parameter κ.