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Some new characterizations of Sobolev spaces

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Abstract

In this paper, we present some new characterizations of Sobolev spaces. Here is a typical result. Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$; we prove that $g \in W^{1,p}(\mathbb{R}^N)$ if and only if

$$\sup_{0<\delta<1} \iint\limits_{\substack{\mathbb{R}^N\,\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Moreover,

$$\lim_{\delta\to 0} \iint\limits_{\substack{\mathbb{R}^N\,\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \quad \forall g \in W^{1,p}\big(\mathbb{R}^N\big),$$

where $K_{N,p}$ is defined by (12).

This result is somewhat related to a characterization of Sobolev spaces due to J. Bourgain, H. Brezis, P. Mironescu (see [J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations, A Volume in Honour of A. Bensoussan's 60th Birthday, IOS Press, 2001, pp. 439–455]). However, the precise connection is not transparent.

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1. Introduction

We first recall a result due to J. Bourgain, H. Brezis, P. Mironescu.

Theorem 1. (J. Bourgain, H. Brezis, P. Mironescu) *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \leqslant C, \quad \forall n \geqslant 1,$$

for some constant $C > 0$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx, \quad \forall g \in L^p\big(\mathbb{R}^N\big),$$

where $K_{N,p}$ is defined by (12). *Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of functions satisfying*

$$\rho_n \geqslant 0, \qquad \rho_n(x) = \rho_n\big(|x|\big),$$

$$\lim_{n\to\infty}\int_{\tau}^{\infty}\rho_n(r)r^{N-1}\,dr = 0, \quad \forall \tau > 0,$$

and

$$\lim_{n\to\infty}\int_{0}^{+\infty}\rho_n(r)r^{N-1}\,dr = 1.$$

Here is a typical example.

Proposition 1. *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\sup_{0<\delta<1}\frac{1}{|\ln\delta|}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop \delta<|x-y|<1}\frac{|g(x)-g(y)|^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Moreover,

$$\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop \delta<|x-y|<1}\frac{|g(x)-g(y)|^p}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx.$$

The reader can find many other interesting examples in [1,3].

In this paper, we present some new characterizations of Sobolev spaces. Our first result is the following.