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Uniqueness properties of solutions of Schrödinger equations

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Abstract

Under suitable assumptions on the potentials V and a, we prove that if $u \in C([0,1], H^1)$ is a solution of the linear Schrödinger equation

$$(i\partial_t + \Delta_x)u = Vu + a \cdot \nabla_x u \ \text{ on } \mathbb{R}^d \times (0,1)$$

and if $u \equiv 0$ in $\{|x| > R\} \times \{0,1\}$ for some $R \geqslant 0$, then $u \equiv 0$ in $\mathbb{R}^d \times [0,1]$. As a consequence, we obtain uniqueness properties of solutions of nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta_x)u = G(x, t, u, \overline{u}, \nabla_x u, \nabla_x \overline{u}) \ \text{ on } \mathbb{R}^d \times (0,1),$$

where G is a suitable nonlinear term. The main ingredient in our proof is a Carleman inequality of the form

$$\|e^{\beta\varphi_\lambda(x_1)} v\|_{L^2_x L^2_t} + \|e^{\beta\varphi_\lambda(x_1)} |\nabla_x v|\,\|_{B^{\infty,2}_x L^2_t} \leqslant \overline{C} \|e^{\beta\varphi_\lambda(x_1)} (i\partial_t + \Delta_x) v\|_{B^{1,2}_x L^2_t}$$

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for any $v \in C(\mathbb{R} : H^1)$ with $v(.,t) \equiv 0$ for $t \notin [0,1]$. In this inequality, $B_x^{\infty,2}$ and $B_x^{1,2}$ are Banach spaces of functions on $\mathbb{R}^d$, and $e^{\beta\varphi_\lambda(x_1)}$ is a suitable weight.

MSC: 35B37

Keywords: Carleman inequalities; Local smoothing; Uniqueness of solutions; Parametrices

1. Introduction

In this paper we study uniqueness properties of solutions of certain nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta_x)u = G(x, t, u, \overline{u}, \nabla_x u, \nabla_x \overline{u}) \text{ on } \mathbb{R}^d \times (0, 1), \tag{1.1}$$

where G is a nonlinear term. We are concerned with the following type of uniqueness question:

Question Q. *Assume that u_1 and u_2 are solutions of* (1.1) *in a suitable function space, and $u_1(x,t) = u_2(x,t)$ for $t \in \{0,1\}$ and $|x| \geqslant R$, for some $R \geqslant 0$. Can we then conclude that $u_1 \equiv u_2$* in $\mathbb{R}^d \times [0,1]$?

This type of uniqueness question seems to originate in control theory and was answered in the affirmative under various assumptions on the nonlinear term $G(x, t, u, \overline{u}, \nabla_x u, \nabla_x \overline{u}) = G(x, t, u, \overline{u})$ (no dependence on the gradient terms $\nabla_x u$ and $\nabla_x \overline{u}$). Zhang [16] used the inverse scattering theory to answer the question Q in the affirmative in the special case $d = 1$, $G = \alpha|u|^2 u$, $\alpha \in \mathbb{R}$, $u_2 \equiv 0$. Bourgain [1] proved uniqueness under analyticity assumptions on the nonlinear term $G = G(u, \overline{u})$, with $u_2 \equiv 0$, and the stronger assumption that u_1 is compactly supported for all $t \in [0,1]$. Kenig et al. [9] answered question Q in the affirmative for sufficiently smooth (in particular, bounded) solutions u_1 and u_2, when $G = G(u, \overline{u})$ satisfies estimates of the form

$$|\nabla G(u, \overline{u})| \leqslant C(|u|^{p_1 - 1} + |u|^{p_2 - 1}), \ 1 < p_1, p_2.$$

The boundedness requirement on the solutions u_1 and u_2 was relaxed to optimal L^p conditions by Ionescu and Kenig [4].

A similar uniqueness question was also considered in the setting of the generalized KdV equation

$$(i^{s-1}\partial_t + \partial_x^s)u = G(x, t, u, \partial_x u, \ldots, \partial_x^{s-1} u) \text{ on } \mathbb{R} \times (0, 1), \ s \geqslant 2$$

by Zhang [15], Bourgain [1] and Kenig et al. [8,10,11], under various assumptions on the nonlinear term G and the solutions u_1 and u_2. A brief description of these results can be found in the introduction of [4].