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Computing the Laplace eigenvalue and level of Maass cusp forms

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ABSTRACT

Let f be a primitive Maass cusp form for a congruence subgroup $\Gamma_0(D) \subset \mathrm{SL}(2,\mathbb{Z})$ and $\lambda_f(n)$ its n-th Fourier coefficient. In this paper it is shown that with knowledge of only finitely many $\lambda_f(n)$ one can often solve for the level D, and in some cases, estimate the Laplace eigenvalue to arbitrarily high precision. This is done by analyzing the resonance and rapid decay of smoothly weighted sums of $\lambda_f(n)e(\alpha n^\beta)$ for $X \leq n \leq 2X$ and any choice of $\alpha \in \mathbb{R}$, and $\beta > 0$. The methods include the Voronoi summation formula, asymptotic expansions of Bessel functions, weighted stationary phase, and computational software. These algorithms manifest the belief that the resonance and rapid decay nature uniquely characterizes the underlying cusp form. They also demonstrate that the Fourier coefficients of a cusp form contain all arithmetic information of the form.

1. Introduction and statement of results

The primary arithmetic information attached to a Maass cusp form is its Laplace eigenvalue. However, in the case of cuspidal Maass forms, the range that these eigenvalues can take is not well-understood. In particular it is unknown if, given a real number r,

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one can prove that there exists a primitive Maass cusp form with Laplace eigenvalue $1/4 + r^2$ (in [9] Hejhal gives a numerical approach which approximates a possible form). Conversely, given the Fourier coefficients of a primitive Maass cusp form f on $\Gamma_0(D)$, it is not clear whether or not one can determine its Laplace eigenvalue. In this paper we show that given only a finite number of Fourier coefficients one can often determine the level D, and then Laplace eigenvalue to arbitrarily high precision. Doing so requires f to have a spectral parameter r which is not too large with respect to the number of known Fourier coefficients of f. This is made precise in the corollaries.

The key to our results will be understanding of the resonance and rapid decay properties of Maass cusp forms. Let f be a primitive Maass cusp form with Fourier coefficients $\lambda_f(n)$. The resonance sum for f (see [15] for background) is given by

$$\sum_{n\geq 1} \lambda_f(n)\phi\left(\frac{n}{X}\right) e(\alpha n^{\beta}) \tag{1}$$

where $\phi \in C_c^\infty((1, 2))$ is a Schwartz function and $\alpha \in \mathbb{R}$ and β, $X > 0$ are real numbers, and $e(x) := \exp(2\pi i x)$.

Sums of this form were first considered in Iwaniec–Luo–Sarnak [11] for f a normalized Hecke eigenform for the full modular group with $\alpha = 2\sqrt{q}$ for $q \in \mathbb{Z}_{>0}$ and $\beta = 1/2$. Later in [15] Ren and Ye investigated this sum in the case when f was a normalized Hecke eigenform for the full modular group, but with no restrictions on α and β. Sun and Wu did the same in [22] for f a Maass cusp form for the full modular group. Ren and Ye then gave resonance results for $\mathrm{SL}(3, \mathbb{Z})$ Maass cusp forms in [16] and [18]. Next, Ren and Ye in [17] and Ernvall-Hytönen–Jääsaari–Vesalainen in [6] considered resonance for $\mathrm{SL}(n, \mathbb{Z})$ Maass cusp forms for $n \geq 2$. Finally, resonance sums were considered in special cases such as Rankin–Selberg products in [5], arithmetic functions relating to primes in [21], and used to derive bounds in terms of the spectral parameter r in [20].

In this paper we take f to be a primitive Maass cusp form for a congruence subgroup $\Gamma_0(D) \subset \mathrm{SL}(2, \mathbb{Z})$. Thus our result extends the family of automorphic forms for which their resonance properties are known. Similar analysis and algorithms can be easily implemented for holomorphic cusp forms for $\Gamma_0(D)$.

In all the above cases estimations of (1) were driven by an interest in understanding of the resonance and rapid decay of this sum. That is, for which choices of α and β does the sum have a large main term in X, and for which choices is it of rapid decay in X. However, in [19] Ren and Ye proposed that resonance could be used in an algorithm to detect the presence of automorphic forms. This view of resonance sums is radically different from that which came before. Traditionally resonance sums are estimated roughly to get a general picture of their behavior. Yet to use them in a computational algorithm one would need to have very precise estimates. The results in this paper grew out of an investigation into the feasibility of implementing the algorithm suggested by Ren and Ye.

The idea of using analytic properties to locate spectral parameters for which Maass cusp forms exist was first considered by Hejhal in [9] (also see [3]). Hejhal's approach was