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Hutchinson–Zaharescu parabolas and Ford circles

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In this note, we study properties of certain parabolas which were recently defined by Hutchinson and Zaharescu in their paper on parabolas infiltrating the Ford circles. We examine this connection between Ford circles and these parabolas. In particular, we characterize the intersection points of these parabolas in terms of tangent Ford circles. Further, we provide a finer analysis of area under the arcs of these parabolas with end points joining the centers of two neighboring Ford circles, than discussed in their paper.

1. Introduction

Certain geometric objects, which have remarkable properties, were recently introduced in [7]. These are parabolas referred to in what follows as Hutchinson–Zaharescu parabolas, defined for each rational number $\frac{a}{q}$ with $(a, q) = 1$ by the quadratic function $P_{a/q} : y = \frac{q^2}{2}\left(x - \frac{a}{q}\right)^2$. Among other things in [7], it is shown that there is a close connection between these parabolas and the more classical objects known as Ford circles.

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Ford circles were introduced by Lester R. Ford in his paper [4] to study the approximation of irrational numbers with rationals. Since then these have been extensively studied, usually exploiting their connection with Farey fractions. See for example [6,8,9] and the references therein. A Ford circle $C_{a/q}$ is a circle tangent to the x-axis at a given point with rational coordinates $(a/q, 0)$ in reduced form, centered at $(a/q, 1/(2q^2))$. Any two Ford circles are either disjoint or tangent to each other. In Theorem 1 of [7], besides proving other relations between the tangency of Ford circles and the points which lie on a given parabola, the following is shown to hold true:

"Given rational numbers r_1 and r_2 in lowest terms and Ford circles C_{r_1} and C_{r_2} of different radii, if the circles C_{r_1} and C_{r_2} are tangent, then the parabolas P_{r_1} and P_{r_2} intersect at exactly two points which are the centers of two Ford circles."

This gives a necessary condition for two Ford circles to be tangent to each other in terms of the points of intersection of two parabolas. In the present paper we complement the results by providing a sufficient condition for this to hold true. We also study the points of intersection which are not centers of Ford circles. In particular we have the following result.

Theorem 1.1. *Let $r_1 = \frac{a_1}{q_1}$ and $r_2 = \frac{a_2}{q_2}$ and P_{r_1} and P_{r_2} denote the Hutchinson–Zaharescu parabolas at r_1 and r_2.*

(1) *A point of intersection of any two parabolas P_{r_1} and P_{r_2} is of the form $\left(\frac{a}{q}, \frac{{m_0}^2}{2q^2}\right)$, where $(a, q) = 1$ and $m_0 \in \mathbb{Z}$.*
(2) *Consider the point $p_0 := \left(\frac{a_0}{q_0}, \frac{m_0^2}{2{q_0}^2}\right)$ where $(a_0, q_0) = 1$ and $m_0 \in \mathbb{Z}$. There exists two Hutchinson–Zaharescu parabolas such that p_0 is a point of intersection of them.*

Remark. Note that part (1) of Theorem 1.1 proves that if two such parabolas intersect at exactly two points which are the centers of two Ford circles then the two Ford circles are tangent to each other.

In the same paper [7] while exploring further connections between these parabolas and Ford circles, the authors find an asymptotic expression for the area under the arcs of a certain set of parabolas. For a fixed positive integer Q, they consider the set of parabolas joining centers of Ford circles that lie on or above the line $y = \frac{1}{2Q^2}$ and look at the area under these arcs. For this, they define a real valued continuous function on the interval $[0, 1]$. The function F_Q is defined on subintervals with end points as two neighboring Farey fractions of order Q. It is given explicitly as

$$F_Q(t) = \frac{1}{2}(q + q')^2 \left(t - \frac{a + a'}{q + q'}\right)^2$$

for all $t \in [a/q, a'/q']$ and a/q, a'/q' are neighboring Farey fractions of order Q. The graph of $F_Q(t)$ coincides with the arc of a Hutchinson–Zaharescu parabola at $\frac{a+a'}{q+q'}$ lying above the interval $[a/q, a'/q']$. Hence the area under these arcs is given by