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On the distribution of numbers related to the divisors of $x^n - 1$

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ABSTRACT

Let $n_1, \cdots, n_r$ be any finite sequence of integers and let S be the set of all natural numbers n for which there exists a divisor $d(x) = 1 + \sum_{i=1}^{deg(d)} c_i x^i$ of $x^n - 1$ such that $c_i = n_i$ for $1 \leq i \leq r$. In this paper we show that the set S has a natural density. Furthermore, we find the value of the natural density of S.

1. Introduction

Cyclotomic polynomials arise naturally as irreducible divisors of $x^n - 1$. The polynomial $x^n - 1$ can be factored in the following way

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{1}$$

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Applying Mobius inversion we get

$$\Phi_n(x) = \prod_{d|n}(x^d - 1)^{\mu(\frac{n}{d})}. \tag{2}$$

The problem of determining the size of maximum coefficient of cyclotomic polynomials has been the subject of the papers [5] and [1]. In [3] Pomerance and Ryan study the size of maximum coefficient of divisors of $x^n - 1$.

It has been proven in [4] that for every finite sequence of integers $(n_i)_{i=1}^r$, there exists $d(x) = 1 + \sum_{i=1}^{deg(d)} c_i x^i$, a divisor of $x^n - 1$ for some $n \in \mathbb{N}$, such that $c_i = n_i$ for $1 \leq i \leq r$. In this paper we investigate the following problem. For a given sequence $(n_i)_{i=1}^r$, let $S(n_1, \cdots, n_r)$ denote the set of all n such that $x^n - 1$ has a divisor $d(x)$ of the form $d(x) = 1 + \sum_{i=1}^{r} n_i x^i + \sum_{i=r+1}^{deg(d)} c_i x^i$. We prove that $S(n_1, \cdots, n_r)$ has a natural density. Observe that if $n \in S(n_1, \cdots, n_r)$ then every multiple of n is in $S(n_1, \cdots, n_r)$.

2. Notation

If $f(x)$ and $g(x)$ are two analytic functions in some neighborhood of 0, we denote $f(x) \equiv g(x) \mod x^{r+1}$ if the coefficients of x^i in the power series of $f(x)$ and $g(x)$ are equal for $0 \leq i \leq r$.

We denote by $\omega(n)$ the number of distinct prime factors of n. Let $\delta(d)$ be 1 if $d \neq 1$ and $\delta(d)$ be -1 otherwise. Note that

$$\Phi_n(x) = \delta(n) \prod_{d|n} \left(1 - x^d\right)^{\mu(\frac{n}{d})}. \tag{3}$$

3. Proof of Main Theorem

We require several lemmas in order to prove that $S(n_1, \cdots, n_r)$ has a natural density.

Lemma 3.1. *For every finite sequence of integers $n_1, \cdots, n_r$ there exists a unique sequence of integers $k_1, \cdots, k_r$ such that*

$$\prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r} n_i x^i \mod x^{r+1}. \tag{4}$$

Proof. The proof is by induction on r. If $r = 1$, using $(1 - x)^{k_1} \equiv 1 - k_1 x(\mod x^2)$, we see that $k_1 = -n_1$ is the unique choice for k_1. Say the lemma holds for $r - 1$, and let A be such that

$$\prod_{i=1}^{r-1}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r-1} n_i x^i + A x^r (\mod x^{r+1}).$$