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On families of linear recurrence relations for the special values of the Riemann zeta function

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ABSTRACT

In this paper, we use the generating function of the Bernoulli polynomials to introduce a number of infinite families of linear recurrence relations for the Riemann zeta function at positive even integer arguments, $\zeta(2n)$.

1. Introduction

The Riemann zeta function or Euler–Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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is probably the most important, fascinating, challenging and mysterious object of modern mathematics, in spite of its utter simplicity. This function is defined over the complex plane and plays a pivotal role in analytic number theory having applications in physics, probability theory, applied statistics and other fields of mathematics. There is an enormous amount of literature on the Riemann zeta function. The classical papers by Abramowitz and Stegun [1], Apostol [3], Berndt [5], Everest, Röttger and T. Ward [9], Ireland and Rosen [10], Murty and Reece [17], and Weil [21] contain excellent additional material related to this article.

Originally the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1.$$

Euler first started to develop the theory of this function and obtained in 1734 the famous formula for even positive zeta values

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}, \tag{1}$$

where n is a positive integer and B_n is the n-th Bernoulli number. There are many proofs of this formula, some of them elementary, see, e.g., [2,4,5,7,18,20,22,23].

Due to Euler's formula, linear recurrence relations for the Bernoulli numbers can be transformed into linear recurrence relations for the Riemann zeta function at even integer arguments. Recently, Merca [16] considered the following relation between Bernoulli numbers, binomial coefficients and powers of 2

$$\sum_{k=0}^{n} \binom{n}{k} 2^k B_k = (2 - 2^n) B_n,$$

and obtained two recurrence relations for the Riemann zeta function with even arguments.

Theorem 1.1. *For $n > 0$,*

$$(-1)^n \frac{\pi^{2n} \cdot n}{(2n+1)!} + \sum_{k=0}^{n-1} (-1)^k \frac{\pi^{2k}}{(2k+1)!} \zeta(2n-2k) = 0. \tag{2}$$

Theorem 1.2. *For $n > 0$,*

$$\zeta(2n) + \frac{2^{2n-1}}{2^{2n}-1} \left((-1)^n \frac{\pi^{2n}}{(2n)!} \cdot \frac{2n-1}{2} + \sum_{k=1}^{n-1} (-1)^k \frac{\pi^{2k}}{(2k)!} \zeta(2n-2k) \right) = 0.$$