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Counting algebraic points of bounded height on projective spaces

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ABSTRACT

We prove new estimates on the number of algebraic points of fixed degree and bounded height on projective spaces over a given number field. These results extend previous works of Wolfgang Schmidt [13], Gao Xia [3] and Martin Widmer [18]. Our approach, based on zeta functions, also gives a new proof of Schanuel's theorem.

Contents

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1. Introduction

1.1. Setting

Let K be a number field. We normalize our valuations on the completions of K in such a way that the product formula

$$\prod_v |x|_v^{d_v} = 1$$

holds for any x in $K^\times$, where d_v is the degree of the extension $K_v/\mathbb{Q}_v$. If x is in K^{n+1}, then we set $||x||_v = \max_i |x_i|_v$ if v is non-archimedean and $||x||_v^2 = \sum_i |x_i|_v^2$ otherwise. This particular choice of norms gives a *height function*

$$H_K(x) = \prod_v ||x||_v^{d_v}$$

on $\mathbb{P}_K^n(K)$, which extends to $\mathbb{P}_K^n(\bar{K})$ via the formula $H_K(x) = H_L(x)^{\frac{1}{[L:K]}}$ where L contains K and x. In this paper we consider the following question: what is the order of magnitude of the cardinality of the set

$$\mathcal{N}(K,n,h,X) = \{x \in \mathbb{P}_K^n(\bar{K}) \mid [K(x):K] = h \text{ and } H_K(x)^h \leq X\}$$

of algebraic points of $\mathbb{P}_K^n$ of fixed degree h and height at most $X^{\frac{1}{h}}$, as X tends to infinity?

1.2. Description of our results

For $h = 1$, and arbitrary n and K, the set $\mathcal{N}(K,n,1,X)$ is the set of K-rational points on $\mathbb{P}_K^n$ of height bounded by X, and asymptotics for its cardinality were found by Schanuel [11] in 1979. He obtained the estimate

$$|\mathcal{N}(K,n,1,X)| \sim A_{K,n,1} X^{n+1}$$

as X tends to infinity, for some constant $A_{K,n,1} > 0$. We will give a new proof of Schanuel's theorem below (see Corollary 3.4.2).

For $n = 1$, and arbitrary h and K, asymptotics for $\mathcal{N}(K,n,h,X)$ have been found by Masser and Vaaler in [9] (see also Le Rudulier's work [8]). More precisely, they obtained

$$|\mathcal{N}(K,1,h,X)| \sim A_{K,1,h} X^{h+1}$$

as X tends to infinity, for some constant $A_{K,1,h} > 0$.

For $K = \mathbb{Q}$, the case $h = 2$ has been handled by Wolfgang Schmidt [13]. We extend his result to arbitrary K: