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### Journal of Number Theory

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# Sums of divisors functions and Bessel function series



Sun Kim

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA

#### ARTICLE INFO

Article history: Received 10 February 2016 Received in revised form 28 February 2016 Accepted 26 June 2016 Available online 3 August 2016 Communicated by David Goss

MSC: primary 11M06 secondary 33C10

Keywords: Ramanujan's lost notebook Bessel functions Circle problem Divisor problem Riesz sum The sum of divisors function

#### ABSTRACT

On page 335 in his lost notebook, Ramanujan recorded without proofs two identities involving finite trigonometric sums and doubly infinite series of Bessel functions. These two identities are intimately connected with the classical *circle* and *divisor problems*, respectively. There are three possible interpretations for the double series of these identities. The first identity has been proved under all three interpretations, and the second under two of them. Furthermore, several analogues of them were established, and they were extended to Riesz sum identities as well. In this paper, we provide analogous Riesz sum identities for the weighted sums of divisors functions, and in particular two of them yield a generalization of the Riesz sum identity for  $r_6(n)$ , where  $r_6(n)$ denotes the number of representations of n as a sum of six squares.

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#### 1. Introduction

On page 335 in his lost notebook [16], Ramanujan recorded two fascinating identities involving doubly infinite series of Bessel functions. It was shown that these identities have

 $\label{eq:http://dx.doi.org/10.1016/j.jnt.2016.06.018} 0022-314 X (© 2016 Elsevier Inc. All rights reserved.$ 

E-mail address: sunkim2@illinois.edu.

connections with the classical *circle and divisor problems* [9,4]. To state Ramanujan's claims, we need to define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases}$$

where, as customary, [x] is the greatest integer less than or equal to x. Let  $J_{\nu}(z)$  denote the ordinary Bessel function of order  $\nu$ , and  $I_{\nu}(z) := -Y_{\nu}(z) + \frac{2}{\pi} \cos(\pi\nu) K_{\nu}(z)$ , where  $Y_{\nu}(z)$  [19, p. 64, eq. (1)] is the Bessel function of the second kind of order  $\nu$  and  $K_{\nu}(z)$ [19, p. 78, eq. (6)] is the modified Bessel function of order  $\nu$ .

**Entry 1.1.** If  $0 < \theta < 1$  and x > 0, then

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4} \cot(\pi\theta) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}.$$

**Entry 1.2.** For x > 0 and  $0 < \theta < 1$ ,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n\theta) = \frac{1}{4} - x \log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}.$$

In [9], Entry 1.1 was proved, but with the order of summation of the double series reversed, and the authors also derived the following corollary from Entry 1.1. Let  $r_2(n)$  denote the number of representations of the positive integer n as a sum of two squares, where representations with different signs and different orders are regarded as distinct representations.

**Corollary 1.3.** For any x > 0,

$$\sum_{0 \le n \le x}' r_2(n) = \pi x + 2\sqrt{x} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}} \right\}.$$
(1.1)

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