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Higher Hickerson formula

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A B S T R A C T

In [11], Hickerson made an explicit formula for Dedekind sums $s(p,q)$ in terms of the continued fraction of p/q. We develop analogous formula for generalized Dedekind sums $s_{i,j}(p,q)$ defined in association with the x^iy^j-coefficient of the Todd power series of the lattice cone in $\mathbb{R}^2$ generated by $(1,0)$ and (p,q). The formula generalizes Hickerson's original one and reduces to Hickerson's for $i=j=1$. In the formula, generalized Dedekind sums are divided into two parts: the integral $s^I_{ij}(p,q)$ and the fractional $s^R_{ij}(p,q)$. We apply the formula to Siegel's formula for partial zeta values at a negative integer and obtain a new expression which involves only $s^I_{ij}(p,q)$ the integral part of generalized Dedekind sums. This formula directly generalizes Meyer's formula for the special value at 0. Using our formula, we present the table of the partial zeta value at $s=-1$ and -2 in more explicit form. Finally, we present another application on the equidistribution property of the fractional parts of

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the graph $\left(\frac{p}{q}, R_{i+j}q^{i+j-2}s_{ij}(p,q)\right)$ for a certain integer R_{i+j} depending on $i+j$.

Contents

1. Introduction

In [11], Hickerson obtained an explicit formula for Dedekind sum $s(p,q)$ in terms of the elements of the continued fraction of p/q, where the Dedekind sum $s(p,q)$ is defined as

$$s(p,q) := \sum_{k=0}^{q-1} \left(\left(\frac{k}{q}\right)\right)\left(\left(\frac{pk}{q}\right)\right).$$

Here $((-))$ denotes the sawtooth function (i.e. for $x \in \mathbb{R}$, $((x)) = x - [x] - \frac{1}{2}$ if $x \notin \mathbb{Z}$ and $((x)) = 0$ otherwise). Hickerson's formula is written as follows:

$$12s(p,q) = \begin{cases} \frac{p-q_{n-1}}{q} + \sum_{i=1}^{n}(-1)^{i+1}a_i & \text{if } n \text{ is even },\\ \frac{p+q_{n-1}}{q} + \sum_{i=1}^{n}(-1)^{i+1}a_i - 3 & \text{if } n \text{ is odd}, \end{cases} \tag{1}$$

where $p_k/q_k = [a_0, \ldots, a_k]$ is the k-th convergent of p/q and a_i are the terms of the continued fraction of p/q:

$$\frac{p}{q} = [a_0, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}} \tag{2}$$

The idea of the proof is successive application of the celebrated reciprocity formula of Dedekind sums: