Journal of Number Theory 170 (2017) 228–249

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Double tails of multiple zeta values

P. Akhilesh [a,b,*,1]

[a] *Harish-Chandra Research Institute, Allahabad, India*
[b] *Institute of Mathematical Sciences, Chennai, India*

A R T I C L E   I N F O

Article history:
Received 9 December 2015
Received in revised form 25 June 2016
Accepted 26 June 2016
Available online 4 August 2016
Communicated by David Goss

MSC:
11M32

Keywords:
Multiple zeta values
Double tails

A B S T R A C T

In this paper we introduce and study double tails of multiple zeta values. We show, in particular, that they satisfy certain recurrence relations and deduce from this a generalization of Euler's classical formula $\zeta(2) = 3\sum_{m=1}^{\infty} m^{-2}\binom{2m}{m}^{-1}$ to all multiple zeta values, as well as a new and very efficient algorithm for computing these values.

1. Introduction

Throughout the paper, $\mathbf{N}$ denotes the set of non-negative integers. A finite sequence $\mathbf{a} = (a_1, \ldots, a_r)$ of positive integers is called *a composition*. The integer r is called *the depth* of $\mathbf{a}$ and the integer $k = a_1 + \ldots + a_r$ *the weight* of $\mathbf{a}$. The composition $\mathbf{a}$ is said to be *admissible* if either $r \geqslant 1$ and $a_1 \geqslant 2$, or $\mathbf{a}$ is the empty composition denoted $\varnothing$.

* Correspondence to: Institute of Mathematical Sciences, Chennai, India.
E-mail address: akhi@imsc.res.in.

[1] The second address is present address.

http://dx.doi.org/10.1016/j.jnt.2016.06.020

To each admissible composition $\mathbf{a} = (a_1, \dots, a_r)$, one associates a real number $\zeta(\mathbf{a})$. It is defined by the convergent series

$$\zeta(\mathbf{a}) = \sum_{n_1 > \dots > n_r > 0} n_1^{-a_1} \dots n_r^{-a_r}, \tag{1}$$

when $r \geqslant 1$, and by $\zeta(\varnothing) = 1$ when $r = 0$. These numbers are called *multiple zeta values* or *Euler–Zagier numbers*.

A *binary word* is by definition a word w constructed on the alphabet $\{0, 1\}$. Its letters are called *bits*. The number of bits of w is called *the weight* of w and denoted by $|w|$. The number of bits of w equal to 1 is called *the depth* of w. To any composition $\mathbf{a} = (a_1, \dots, a_r)$, one associates the binary word

$$\mathbf{w}(\mathbf{a}) = \{0\}_{a_1 - 1} 1 \dots \{0\}_{a_r - 1} 1 \tag{2}$$

where for each integer $u \geqslant 0$, $\{0\}_u$ denotes the binary word consisting of u bits equal to 0, and where $\mathbf{w}(\mathbf{a})$ is the empty word if $\mathbf{a}$ is the empty composition. The weight of $\mathbf{w}(\mathbf{a})$ is equal to the weight of $\mathbf{a}$ and its depth to the depth of $\mathbf{a}$.

We shall denote by W the set of binary words. When $\varepsilon, \varepsilon' \in \{0, 1\}$, ${}_\varepsilon\mathrm{W}$ and $\mathrm{W}_{\varepsilon'}$ denote the sets of binary words starting by ε and ending by ε' respectively, and ${}_\varepsilon\mathrm{W}_{\varepsilon'}$ their intersection.

The map $\mathbf{w}$ is a bijection from the set of compositions onto the set of binary words not ending by 0. Non-empty compositions correspond to words in W_1, and non-empty admissible compositions to words in ${}_0\mathrm{W}_1$. Therefore a binary word will be called *admissible* if either it belongs to ${}_0\mathrm{W}_1$, or it is empty.

Maxim Kontsevich has discovered that for each admissible composition $\mathbf{a}$, the multiple zeta value $\zeta(\mathbf{a})$ can be written as an iterated integral. More precisely, if $w = \varepsilon_1 \dots \varepsilon_k$ denotes the associated binary word $\mathbf{w}(\mathbf{a})$, we have

$$\zeta(\mathbf{a}) = \mathrm{It}\int_0^1 (\omega_{\varepsilon_1}, \dots, \omega_{\varepsilon_k}) = \int_{1 > t_1 > \dots > t_k > 0} f_{\varepsilon_1}(t_1) \dots f_{\varepsilon_k}(t_k) dt_1 \dots dt_k \tag{3}$$

where $\omega_i = f_i(t)dt$, with $f_0(t) = \frac{1}{t}$ and $f_1(t) = \frac{1}{1-t}$. We therefore often write this number $\zeta(w)$ instead of $\zeta(\mathbf{a})$.

Let $w = \varepsilon_1 \dots \varepsilon_k$ be a binary word. Its *dual word* is defined to be $\overline{w} = \bar{\varepsilon}_k \dots \bar{\varepsilon}_1$, where $\bar{0} = 1$ and $\bar{1} = 0$. When w is admissible, so is $\overline{w}$. We can therefore define the *dual composition* of an admissible composition $\mathbf{a}$ to be the admissible composition $\bar{\mathbf{a}}$ such that $\mathbf{w}(\bar{\mathbf{a}})$ is dual to $\mathbf{w}(\mathbf{a})$. When $\mathbf{a}$ has weight k and depth r, $\bar{\mathbf{a}}$ has weight k and depth $k - r$.

By the change of variables $t_i \mapsto 1 - t_{k+1-i}$ in the integral (3), one gets the following *duality relation*: for any admissible composition $\mathbf{a}$, we have

$$\zeta(\mathbf{a}) = \zeta(\bar{\mathbf{a}}). \tag{4}$$