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Multiplicative relations for Fourier coefficients of degree 2 Siegel eigenforms

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We prove multiplicative relations between certain Fourier coefficients of degree 2 Siegel eigenforms. These relations are analogous to those for elliptic eigenforms. We also provide two sets of formulas for the eigenvalues of degree 2 Siegel eigenforms. The first evaluates the eigenvalues in terms of the form's Fourier coefficients, in the case $a(I) \neq 0$. The second expresses the eigenvalues of index p and p^2, for p prime, solely in terms of p and k, the weight of the form, in the case $a(0) \neq 0$. From this latter case, we give simple expressions for the eigenvalues associated to degree 2 Siegel Eisenstein series.

1. Introduction and statement of main results

The theory of Hecke operators provides us with many of the fundamental results about the spaces of elliptic modular forms. For example, the space of elliptic modular forms, of a given weight, has a basis of Hecke eigenforms which have multiplicative Fourier coefficients.

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Hecke theory has been extended to Siegel modular forms, with the work of Andrianov at its core (see for example [1–3]), but in some respects the results aren't as satisfying. While the spaces of Siegel modular forms have a basis of eigenforms, and Andrianov provides us with a comprehensive structure for the relationship between the eigenvalues and Fourier coefficients of degree 2 eigenforms, we do not get simple multiplicative relations between the Fourier coefficients, as exists in the elliptic case. The main purpose of this paper is to prove, using the results of Andrianov, that simple multiplicative relations, which are analogous to the elliptic case, do exist between certain Fourier coefficients of degree 2 Siegel eigenforms.

Let f be an elliptic eigenform of weight $k \geq 1$ on the full modular group, with Fourier expansion $f(z) = \sum_{n\geq 0} a(n)q^n$, with $q := e^{2\pi i z}$. Then its Fourier coefficients satisfy the following properties [4,6]:

(1) If $(1) = 0$, then $a(m) = 0$ for all $m \in \mathbb{Z}^+$.
(2) $a(1)\, a(mn) = a(m)\, a(n)$ when $\gcd(m, n) = 1$.
(3) $a(1)\, a(p^{r+1}) = a(p)\, a(p^r) - p^{k-1}\, a(1)\, a(p^{r-1})$, for all p prime and $r \geq 1$.

If f is non-zero, then property (1) ensures that $a(1) \neq 0$, and we can normalize f by setting $a(1) = 1$. In which case, we can drop the $a(1)$ factors in properties (2) and (3).

The main result of this paper provides analogous properties for degree 2 Siegel eigenforms. Let $M_k^2(\Gamma)$ denote the space of Siegel modular forms of degree 2 and weight k on the full modular group. Let I denote the 2×2 identity matrix.

Theorem 1.1. *Let $F(Z) = \sum_{N\geq 0} a(N)\, exp(2\pi i\, \mathrm{Tr}(NZ)) \in M_k^2(\Gamma)$ be an eigenform.*

(1) *If $a(I) = 0$, then $a(mI) = 0$ for all $m \in \mathbb{Z}^+$.*
(2) $a(I)\; a(mnI) = a(mI)\; a(nI)$ *when* $\gcd(m, n) = 1$.
(3) $$a(I)\; a\left(p^{r+1}I\right) = a(pI)\; a(p^rI) - p^{2k-3}\, a(I)\; a\left(p^{r-1}I\right) - p^{k-2}\, a(I) \left[2\, a\begin{pmatrix} p^{r-1} & 0 \\ 0 & p^{r+1}\end{pmatrix} + (1+(-1)^k) \sum_{\substack{u=1 \\ u^2 \not\equiv -1 \pmod p}}^{\frac{p-1}{2}} a\left(p^r \begin{pmatrix} (1+u^2)p^{-1} & u \\ u & p\end{pmatrix}\right)\right],$$
for all p prime and $r \geq 1$, and where the last sum is vacuous in the case $p = 2$.

If $a(I) \neq 0$, we can normalize F by setting $a(I) = 1$. In this case we can remove the $a(I)$ factors from properties (2) and (3) of Theorem 1.1. Unfortunately, property (1) of Theorem 1.1 is not sufficient to ensure $a(I) \neq 0$ for non-zero eigenforms, as happens in the case of elliptic eigenforms. In fact, if the weight k is odd then $a(I) = 0$, which we will see in Section 2. However, the following classes of degree 2 Siegel eigenforms all have $a(I) \neq 0$: Siegel Eisenstein series, E_k, which have even weight $k \geq 4$ (see Section 2); the unique cusp eigenforms of weights 10 and 12, often denoted χ_{10} and χ_{12}, which along with the Eisenstein series E_4 and E_6 generate the graded ring of even weight degree 2 Siegel modular forms [9]; and Klingen Eisenstein series generated from elliptic eigenforms [12].