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Multiplicative Diophantine equations with factors from different Lucas sequences

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ABSTRACT

Let $k \geq 2$ and $\{u_n^{(1)}\}_{n\geq 0}, \ldots, \{u_n^{(k)}\}_{n\geq 0}$ be k different nondegenerate binary recurrent sequences of integers. In this paper, we show that under certain conditions, there are only finitely many of k-tuples of the form $(u_{n_1}^{(1)}, \ldots, u_{n_k}^{(k)})$ whose contents are multiplicatively dependent. Furthermore, we determine all such instances when $k = 3$ and the three sequences are the sequence of Fibonacci numbers, Pell numbers, and base 10 repunits, respectively.

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1. Introduction

In this paper, we study multiplicative relations among members of different binary recurrent sequences. Recall that an integral binary recurrent sequence $\mathbf{u} = \{u_n\}_{n\geq 0}$ is a sequence satisfying $u_{n+2} = pu_{n+1} + qu_n$ for all $n \geq 0$, with $u_0,\ u_1,\ p,\ q \in \mathbb{Z}$. Assuming $p^2 + 4q \neq 0$, the Binet formula is

$$u_n = c\alpha^n + d\beta^n \qquad \text{for all} \qquad n \geq 0,$$

where α, β are the roots of $x^2 - px - q = 0$ and c, d are some numbers which can be computed in terms of u_0, u_1, α, β. One of the very active areas of research concerns the arithmetic properties of the numbers u_n. For example, asking when is u_n a square, a cube or a perfect power of some positive exponent of some other integer is equivalent to asking when are all the exponents of the primes appearing in the factorization of u_n even, or multiples of 3, or have a greatest common divisor larger than 1. For several classical sequences like Fibonacci numbers F_n (corresponding to $p = q = 1$, $F_0 = 0$, $F_1 = 1$), or Pell numbers P_n (corresponding to $p = 2$, $q = 1$, $P_0 = 0$, $P_1 = 1$) all perfect powers which belong to these sequences are known. More about this in Section 5. One question that has received a lot of attention concerns the largest prime factor of u_n. Putting $P(m)$ for the largest prime factor of the positive integer m, an intriguing question is to find good lower bounds on $P(u_n)$. In the simplest case when $u_n = 2^n - 1$ (corresponding to $p = 3$, $q = -2$, $u_0 = 0$, $u_1 = 1$), Schinzel [16] showed that $P(2^n - 1) \geq 2n + 1$ for all $n > 12$. For large n, Stewart [18] did much better and proved that $P(2^n - 1) > n \exp(\log n/(104 \log\log n))$ thus confirming a conjecture of Erdős to the effect that $\lim_{n\to\infty} P(2^n - 1)/n = \infty$. Under the *abc*-conjecture, Murty and Wong [15] proved that $P(2^n - 1) > n^{2-\varepsilon}$ holds for all $\varepsilon > 0$ as long as n is sufficiently large in a way depending on ε. Murata and Pomerance [14] write that "It is perhaps reasonable to conjecture that $P(2^n - 1) > n^K$ for all sufficiently large K and all sufficiently large n depending on K, or maybe even $P(2^n - 1) > 2^{n/\log n}$ for all sufficiently large n, but clearly we are very far from proving such assertions".

Now let us take a look at what consequences one could derive if the opposite of what we believe to be true actually holds, namely that $P(u_n)$ is "small" for "many" values of n. Well, if one quantifies appropriately the above assertion then one would get as a consequence that there are many numbers among the u_n's all with small primes, so maybe one should then get that there are many multiplicative relations among such numbers. While this chain of thought might seem that it might not give good results, this principle was successful in at least two instances. In the first instance, it was applied by Wirsing and Schinzel in [17] to show that some partition numbers $p(n)$ have large prime factors. In a second instance, it was applied by Luca, Pizarro-Madariaga and Pomerance in [11] to give an explicit lower bound on the number or irregular primes $p \leq x$. Thus, it could be of interest to study multiplicative relations among members of linear recurrences. Even this problem is not new. Luca and Ziegler [12] studied the Diophantine equation