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A new general asymptotic formula and inequalities involving the volume of the unit ball

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ABSTRACT

In this paper, based on some early works, we establish a general continued fraction approximation for the nth root of the volume of the unit n-dimensional ball. Then related inequalities are given. Finally, for demonstrating the superiority of our new estimates and inequalities, we present some numerical computations.

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1. Introduction

In the recent past, some authors presented many asymptotic series and inequalities about the volume of the unit ball in $\mathbb{R}^n$ (see, e.g., [5]) for every integer $n \geq 1$:

$$\Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \tag{1.1}$$

where Γ denotes Euler's gamma function.

According to [5], the sequence itself is not monotonic and it attains its maximum at $n = 5$. But as it was proved by Anderson in 1989 [4], $\Omega_n^{1/n}$ strictly decreases with $\lim_{n\to\infty} \Omega_n^{1/n} = 0$. In 1997, Anderson and Qiu [3] showed that $\Omega_n^{1/(n \ln n)}$ is also strictly decreasing with $\lim_{n\to\infty} \Omega_n^{1/(n \ln n)} = e^{-1/2}$.

The monotonicity theorems provided in [4] and [7] lead to the following inequalities for every integer $n \geq 1$

$$\Omega_{n+1}^{\frac{n}{n+1}} < \Omega_n, \tag{1.2}$$

$$1 < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n}. \tag{1.3}$$

Motivated by (1.2) and (1.3), Alzer [2] proved that for all integers $n \geq 1$,

$$a\Omega_{n+1}^{\frac{n}{n+1}} \leq \Omega_n < b\Omega_{n+1}^{\frac{n}{n+1}}, \tag{1.4}$$

with the best possible constants $a = 2/\sqrt{\pi} = 1.12837\cdots$ and $b = \sqrt{e} = 1.64872\cdots$.

As an improvement of (1.4), Chen [6] provided the proof of the following double inequalities for every integer $n \geq 1$:

$$\frac{1}{\sqrt{\pi(n+a)}}\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \leq \Omega_n < \frac{1}{\sqrt{\pi(n+b)}}\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}}, \tag{1.5}$$

where $a = e/2 - 1$, $b = 1/3$.

Furthermore, as it was proved by Mortici [12,13], the inequalities

$$\begin{aligned} &-\frac{1}{2}\ln n + \frac{1}{2}(1+\ln 2\pi) - \frac{1}{2n}\ln n\pi - \lambda(n) \\ &\quad < \frac{1}{n}\ln \Omega_n < -\frac{1}{2}\ln n + \frac{1}{2}(1+\ln 2\pi) - \frac{1}{2n}\ln n\pi - \mu(n) \end{aligned} \tag{1.6}$$

hold true for every integer $n \geq 1$, where

$$\mu(n) = \frac{1}{6n^2} - \frac{1}{45n^4} + \frac{8}{315n^6} - \frac{8}{105n^8} \tag{1.7}$$

and