Journal of Number Theory 170 (2017) 315–346

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

"Strange" combinatorial quantum modular forms

Amanda Folsom *, Caleb Ki, Yen Nhi Truong Vu, Bowen Yang

A R T I C L E   I N F O

Article history:
Received 20 October 2015
Received in revised form 7 June 2016
Accepted 8 June 2016
Available online 3 August 2016
Communicated by Ahmad El-Guindy

Keywords:
q-Series
q-Hypergeometric series
Basic hypergeometric series
Modular forms
Quantum modular forms
Unimodal sequences
Strongly unimodal sequences

A B S T R A C T

Motivated by the problem of finding explicit q-hypergeometric series which give rise to quantum modular forms, we define a natural generalization of Kontsevich's "strange" function. We prove that our generalized strange function can be used to produce infinite families of quantum modular forms. We do not use the theory of mock modular forms to do so. Moreover, we show how our generalized strange function relates to the generating function for ranks of strongly unimodal sequences both polynomially, and when specialized on certain open sets in $\mathbb{C}$. As corollaries, we reinterpret a theorem due to Folsom–Ono–Rhoades on Ramanujan's radial limits of mock theta functions in terms of our generalized strange function, and establish a related Hecke-type identity.

1. Introduction and statement of results

1.1. Background and motivation

Quantum modular forms have been a topic of recent interest. Loosely speaking, as defined by Zagier [18], a quantum modular form is a complex-valued function that exhibits modular-like transformation properties on the rational numbers, as opposed to the upper-half of the complex plane. To be more precise, a *weight k quantum modular form*

* Corresponding author.
E-mail address: afolsom@amherst.edu (A. Folsom).

http://dx.doi.org/10.1016/j.jnt.2016.06.005

($k \in \frac{1}{2}\mathbb{Z}$) is a complex-valued function f on $\mathbb{Q}$ or possibly $\mathbb{P}^1(\mathbb{Q}) \setminus S$ for some appropriate set S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is an appropriate subgroup, the function

$$h_\gamma(x) = h_{f,\gamma}(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a 'suitable' property of continuity or analyticity. The $\epsilon(\gamma)$ are appropriate complex numbers, such as those that arise naturally in the theory of half-integral weight modular forms. Here, we have modified Zagier's original definition as in [7] to allow half-integral weights k, subgroups of $\mathrm{SL}_2(\mathbb{Z})$, and multiplier systems $\epsilon(\gamma)$, in accordance with the theory of ordinary modular forms. Zagier's definition, in particular the continuity or analyticity requirement of the "error to modularity" $h_\gamma(x)$, is intentionally vague, so that it may encompass many diverse, interesting, examples.

Among Zagier's pioneering first examples of quantum modular forms is the function $\phi(x) := e(x/24)F(e(x))$ ($e(z) := e^{2\pi i z}$), where $x \in \mathbb{Q} \setminus \{0\}$, and the function $F(q)$ is the "strange" function

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n$$

(where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}$, and $(a;q)_0 := 1$) originally studied by Kontsevich [18]. One "strange" aspect of the function $F(q)$ is that it converges on no open subset of $\mathbb{C}$, only when $q = \zeta_k^h := e(h/k)$ ($k \in \mathbb{N}$, $h \in \mathbb{Z}$) is a root of unity. In [18], Zagier proves that the normalized strange function $\phi(x)$ in fact possesses some beautiful analytic properties, which we paraphrase in the following theorem.

Theorem *(Zagier, [18]). For $x \in \mathbb{Q} \setminus \{0\}$, we have that $\phi(x)$ is quantum modular form of weight $3/2$ with respect to the group $\mathrm{SL}_2(\mathbb{Z})$. In particular, $h_{\gamma,\phi}(x)$ is a real analytic function.*

Perhaps surprisingly, $F(q)$ has also been connected to a certain function $U(1;q)$ which is of independent interest for its combinatorial properties, and which was also shown in [5] to be both mock modular and quantum modular. (For more on mock modular forms and their numerous applications in recent years, we refer the interested reader to the surveys by Ono [12] and Zagier [17].) To describe this connection more precisely, we introduce some combinatorial functions. A sequence $\{a_j\}_{j=1}^s$ of integers is called a *strongly unimodal sequence of size* n if there exists some integer r such that $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots > a_s > 0$, and $a_1 + a_2 + \cdots + a_s = n$. Analogous to the notion of the rank of an integer partition, one also has the notion of the rank of a strongly unimodal sequence; in terms of the definition given above, the *rank* of the strongly unimodal sequence $\{a_j\}_{j=1}^s$ is defined to be $s - 2r + 1$, the number of terms after the maximal term in the sequence minus the number of terms that precede it. It is