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Diophantine inequalities involving a prime and an almost-prime

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ABSTRACT

We prove that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{131}},$$

where λ_0 is an arbitrary real number and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 \neq 0$ and $0 > \frac{\lambda_1}{\lambda_2}$ not in $\mathbb{Q}$. This improves a result by Harman.

1. Introduction

In Diophantine Approximation, a classical theorem of Kronecker ([2], Theorem 440) indicates that there are infinitely many solutions in positive integers n_1, n_2 of

$$|\lambda_0 + \lambda_1 n_1 + \lambda_2 n_2| < 3\left(\max\left\{\frac{n_1}{\lambda_2}, \frac{n_2}{\lambda_1}\right\}\right)^{-1},$$

where $\frac{\lambda_1}{\lambda_2}$ is irrational and λ_0 is an arbitrary real number.

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The case where n_1 and n_2 are both primes is of great interest and remains open to date ([6,7]). The first approximation in this direction has been given by Vaughan [8] who proved that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_4| < p^{-1/600000},$$

where and henceforth in this paper the letter p denotes a prime and P_r a number with at most r prime factors. Harman [4] proved that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\tau}, \tag{1}$$

with $\tau = \frac{1}{300}$.

In this paper, we will improve Harman's result by showing that in (1) one can actually take $\tau = \frac{1}{131}$. The main result of this paper will be the following theorem.

Theorem 1. *For $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ with $\frac{\lambda_1}{\lambda_2}$ both negative and irrational, there are infinitely many solutions of*

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{131}}.$$

2. Notation and outline of the method

2.1. Notation

We shall use η and ε for arbitrary small positive numbers; and sometimes they may be slightly different in context just for simplicity.

We write $\lfloor x \rfloor$ for the largest integer not exceeding x. We write $\|x\|$ for the distance from x to a nearest integer and $\lceil x \rfloor$ for the nearest integer to x when $\|x\| \neq \frac{1}{2}$. Clearly we may assume that $\lambda_1 > 0$ and $\lambda_2 = -1$. Let $\frac{a}{q}$ be a convergence to the continued fraction for λ_1 and assume q to be quite large in terms of λ_0, λ_1 and λ_1^{-1}; let X be a large number such that $q \asymp X^{\frac{1}{3}+\rho+\eta}$. Trivially, one can write $\lambda_0 = \frac{b}{q} + \gamma$ with $|\gamma| < \frac{1}{q}$.

As in [4], we assume that q is so large that $\min\{\frac{a}{q}, \frac{q}{a}\} > X^{-\frac{\rho}{4}}$ and $aX + b < qX^{1+\frac{\eta}{4}}$. In this paper, p, $p_i, i = 1, 2, \ldots$ represent primes; $\sum^\flat$ indicates that the summation is only over square-free numbers. For convenience, we shall denote

$$e(x) := \exp(2\pi i x), \qquad \xi := X^{-\rho}, \quad \text{where } \rho \text{ is a positive number};$$

$$P(z) := \prod_{p \le z} p, \qquad Y := \lfloor 3\xi^{-1} X^{\eta} \rfloor.$$

2.2. The weighted sieve

Essentially, if we use the same method as in [4] but with a parameterized weight to optimize the result, we will obtain that $\tau = \frac{1}{147}$ is admissible as mentioned in Section 6.