Journal of Number Theory 170 (2017) 368–389

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Formulae for the Frobenius number in three variables

Amitabha Tripathi

Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi 110016, India

A R T I C L E I N F O

Article history:
Received 19 October 2015
Received in revised form 3 April 2016
Accepted 2 May 2016
Available online 18 August 2016
Communicated by S. Miller

MSC:
11D07
11D04

Keywords:
Representable
Frobenius number

A B S T R A C T

Text. For positive integers a, b, c that are coprime, the Frobenius number of a, b, c, denoted by $\mathtt{g}(a, b, c)$, is the largest integer that is not expressible by the form $ax + by + cz$ with x, y, z nonnegative integers. We give *exact* formulae for $\mathtt{g}(a, b, c)$ that covers *all* cases of a, b, c.

Video. For a video summary of this paper, please visit https://youtu.be/dv0GSy2MGzw.

1. Introduction

The `Frobenius Problem` (`FP`) is to determine the largest positive integer that is not representable as a nonnegative integer combination of given positive integers that are coprime. Due to an obvious connection with supplying change in terms of coins of certain fixed denominations, the Frobenius problem is also known as the `Coin Exchange Problem` or as the `Money Changing Problem`. More formally, given positive integers

E-mail address: atripath@maths.iitd.ac.in.

http://dx.doi.org/10.1016/j.jnt.2016.05.027
0022-314X/

$a_1, \dots, a_n$, with $\gcd(a_1, \dots, a_n) = 1$, it is well known and not hard to show that for all sufficiently large N the equation

$$a_1x_1 + \cdots + a_nx_n = N \tag{1}$$

has a solution with nonnegative integers $x_1, \dots, x_n$. The `Frobenius number` $\mathtt{g}(a_1, \dots, a_n)$ is the *largest* integer N such that (1) has no solution in nonnegative integers. Although the origin of the problem is attributed to Sylvester [26], who showed that $\mathtt{g}(a_1, a_2) = a_1a_2 - a_1 - a_2$, an apparent reason for associating the name of Frobenius with this problem is possibly due to the fact that he was largely instrumental in popularizing this problem in his lectures. The Frobenius problem has a rich and long history, with several applications and extensions, and connections to several areas of research. A comprehensive survey covering all aspects of the problem can be found in [18]; also see [8].

Exact determination of the Frobenius number is a difficult problem in general. Brauer [3] found the Frobenius number for consecutive integers, Roberts [20] extended this result to numbers in arithmetic progression (see also [1,29,34]), and Selmer [24] further generalized this to the determination of $\mathtt{g}(a, ha+d, ha+2d, \dots, ha+nd)$ (see also [31]). There are only a few other cases where the Frobenius number has been exactly determined for any n variables; refer to [18] for other instances. In the absence of exact results, research on the Frobenius problem has often been focused on sharpening bounds on the Frobenius number and on algorithmic aspects. Although running time of these algorithms is superpolynomial, Kannan [15] gave a method that solved the Frobenius problem in polynomial time for *fixed* number of variables using the concept of covering radius, and Ramírez Alfonsín [17] showed that the problem is `NP-hard` under Turing reduction.

The purpose of this article is to give exact results for the Frobenius number $\mathtt{g}(a_1, a_2, a_3)$ in all cases. Most of the results in this article appeared in the author's thesis [28], but were not communicated earlier. Although the Frobenius number $\mathtt{g}(a_1, a_2)$ is easy to determine, exact formulae for $\mathtt{g}(a_1, a_2, a_3)$ for all choices of the variables were not previously known and results concerning this were limited to algorithms, bounds and exact results in some special cases.

1.1. A brief overview

We divide our article into three sections. We begin with a brief introduction to the `FP` in Section 1. In Section 2, we give a historical perspective to the special case of the `FP` in three variables, and cite two crucial results (Theorems 1 and 2) we use to obtain our formulae. Section 3 contains the formulae for $\mathtt{g}(a, b, c)$. For convenience, we have subdivided this into six subsections. We give two independent sets of formulae, each of which covers all cases of a, b, c. Both sets include the results given in Lemmas 1 and 2; additionally, one set of results is given by Theorems 3 and 5, while the other set is given by results in Theorems 4 and 6. The subcases covered by Theorems 3 and 4 give