Journal of Number Theory 172 (2017) 1–20

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the theorem of Davenport and generalized Dedekind sums

Bence Borda

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

A R T I C L E   I N F O

Article history:
Received 23 June 2016
Received in revised form 22 August 2016
Accepted 24 August 2016
Available online 8 October 2016
Communicated by D. Goss

MSC:
11K38
11F20

Keywords:
Discrepancy
Generalized Dedekind sum

A B S T R A C T

A symmetrized lattice of $2n$ points in terms of an irrational real number α is considered in the unit square, as in the theorem of Davenport. If α is a quadratic irrational, the square of the L^2 discrepancy is found to be $c(\alpha)\log n + O(\log\log n)$ for a computable positive constant $c(\alpha)$. For the golden ratio φ, the value $\sqrt{c(\varphi)\log n}$ yields the smallest L^2 discrepancy of any sequence of explicitly constructed finite point sets in the unit square. If the partial quotients a_k of α grow at most polynomially fast, the L^2 discrepancy is found in terms of a_k up to an explicitly bounded error term. It is also shown that certain generalized Dedekind sums can be approximated using the same methods. For a special generalized Dedekind sum with arguments a, b an asymptotic formula in terms of the partial quotients of $\frac{a}{b}$ is proved.

1. Introduction

Consider an arbitrary finite set $A \subset [0,1]^2$ in the unit square. For any $x, y \in [0,1]$ let

$$S_A(x,y) = |A \cap ([0,x) \times [0,y))| \tag{1}$$

E-mail address: bordabence85@gmail.com.

http://dx.doi.org/10.1016/j.jnt.2016.08.016

denote the number of elements of A in the rectangle $[0,x) \times [0,y)$. A classical result of K. Roth [8] in the theory of discrepancy is that for any finite set $A \subset [0,1]^2$ we have

$$\int_0^1 \int_0^1 \left(S_A(x,y) - |A|xy\right)^2 \, \mathrm{d}x \, \mathrm{d}y > C \cdot \log |A| \tag{2}$$

for some universal constant $C > 0$. The square root of the left hand side of (2) is sometimes called the mean square discrepancy, or the L^2 discrepancy of the set A.

Several constructions for the set A show that (2) is best possible up to a constant factor, the first of which is due to H. Davenport. For a positive integer n and an irrational real number α consider the set

$$A = A(n,\alpha) = \left\{ \left(\{\pm k\alpha\}, \frac{k}{n} \right) : 1 \le k \le n \right\} \tag{3}$$

of $2n$ points, where $\{x\}$ denotes the fractional part of x. The theorem of Davenport [5] states that if α is badly approximable, i.e. $\inf_{m>0} m \|m\alpha\| > 0$, where $\|x\|$ denotes the distance of x from the nearest integer, then for A as in (3) we have

$$\int_0^1 \int_0^1 \left(S_A(x,y) - |A|xy\right)^2 \, \mathrm{d}x \, \mathrm{d}y < C(\alpha) \cdot \log |A| \tag{4}$$

for some positive constant $C(\alpha)$ depending only on α.

The purpose of this paper is to find the precise order of magnitude of the left hand side of (4), where A is as in (3). We will work with a weaker assumption than α being badly approximable, however: we will assume that the continued fraction representation $\alpha = [a_0; a_1, a_2, \dots]$ satisfies $a_k = O(k^d)$ for some constant $d \ge 0$. Note that α is badly approximable if and only if this condition holds with $d = 0$. The motivation for this generality comes from the fact that the partial quotients of Euler's number e satisfy $a_k = O(k)$. In fact, there is a class of transcendental numbers related to Euler's number e, including e.g. $e^{\frac{2}{n}}$ for every positive integer n, the partial quotients of which satisfy the same condition. Since there are very few classes of irrational numbers the continued fraction representations of which are explicitly known, we wanted our results to hold for as many of them as possible.

2. The main term of the L^2 discrepancy

The original proof of Davenport [5] of (4) heavily uses the properties of the sequence $\|m\alpha\|$. The first step toward finding the precise order of magnitude of the left hand side of (4) is to isolate its dependence on $\|m\alpha\|$ as follows.