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Sign changes of Hecke eigenvalue of primitive cusp forms ☆

Zhao Xu

School of Mathematics, Shandong University, Jinan, Shandong 250100 China

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ABSTRACT

Under a slightly better zero-free region of the corresponding modular L-function, we get a very small bound for the size of first sign change of Hecke eigenvalues for the classical modular forms. When it comes to the prime argument, we derive, for almost all primitive forms, a small bound for the first sign change on prime numbers.

1. Introduction

Let $H_k^*(N)$ be the finite set of all primitive holomorphic cusp forms of weight k for $\Gamma_0(N)$, where $k \geqslant 2$ is an even integer and $N \geqslant 1$ is an integer. We restrict the Nebentypus to be trivial such that all the Hecke eigenvalues are real. For any $f \in H_k^*(N)$,

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E-mail address: zxu@sdu.edu.cn.

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denote by $\lambda_f(n)$ its n-th normalized Hecke eigenvalue such that $\lambda_f(1) = 1$. One central problem in modular form theory is to study the sign change of Hecke eigenvalues $\lambda_f(n)$. In [5], Knopp–Kohnen–Pribitkin proved that $\lambda_f(n)$ has infinitely many sign changes. But the question is that of how soon one can see that the first negative Hecke eigenvalue in terms of the conductor is $Q = k^2N$. This problem actually can be viewed as some kind of Linnik's problem in nature.

We denote by n_f the smallest integer such that $\lambda_f(n_f) < 0$ and $(n_f, N) = 1$. The first result is due to Kohnen–Sengupta [6], which states that

$$n_f \ll kN \exp\left(c\sqrt{\frac{\log N}{\log\log(3N)}}\right)(\log k)^{27}.$$

Then they joined forces with Iwaniec in [3] to show that

$$n_f \ll Q^{29/60}.$$

In 2010, Kowalski–Lau–Soundararajan–Wu [7] introduced some new ingredients and derived that

$$n_f \ll Q^{9/20},$$

where the implied constant is absolute. Consequently, Matomäki [11] pushed their idea to limit, getting

$$n_f \ll Q^{3/8}$$

which is the best known result by now.

There are many sign change results of other automorphic forms, see [8–10,12,13], etc.

However, the upper bound of n_f is probably far from optimal. Indeed, it can be connected with the zero-free region of the attached L-function $L(s, f)$. And one can show that under the Grand Riemann Hypothesis we have

$$n_f \ll \log^2 Q,$$

where the implied constant is absolute. The current zero-free region takes the form

$$\sigma > 1 - \frac{c}{\log(Q(|t| + 2))}, \tag{1.1}$$

where $c > 0$ is an absolute constant, and $s = \sigma + it$. This result probably does nothing to n_f. Surprisingly, we can make a big improvement under the assumption of a little better zero-free region. Since GRH is widely believed to be true, this is reasonable. To this end, we have the following result.