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The parity theorem for multiple polylogarithms

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ABSTRACT

We generalize the well-known parity theorem for multiple zeta values (MZV) to functional equations of multiple polylogarithms (MPL). This reproves the parity theorem for MZV with an additional integrality statement, and also provides parity theorems for special values of MPL at roots of unity (also known as coloured MZV). We give explicit formulas in depths 2 and 3 and provide a computer program to compute the functional equations.

1. Introduction

Multiple zeta values (MZV) are defined for integers $\boldsymbol{n} \in \mathbb{N}^d$ with $n_d > 1$ as

$$\zeta(\boldsymbol{n}) = \zeta(n_1, \ldots, n_d) := \sum_{0<k_1<\cdots<k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}} \tag{1.1}$$

where d is called *depth* and $|\boldsymbol{n}| = n_1 + \cdots + n_d$ is the *weight* [19,35]. We set $\mathbb{N}^0 := \{\emptyset\}$ and $\zeta(\emptyset) := 1$ in weight $|\emptyset| := 0$ and we write

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$$\mathcal{Z}_w^d := \mathrm{lin}_{\mathbb{Q}} \left\{ \zeta^k(2)\zeta(\boldsymbol{n}) \colon \ \boldsymbol{n} \in \mathbb{N}^r, k \in \mathbb{N}_0, n_r > 1, |\boldsymbol{n}| + 2k = w, r \leq d \right\} \tag{1.2}$$

for all rational linear combinations of MZV with weight w and depth at most d. In our convention all powers of $\zeta(2)$ have depth zero, hence $\mathcal{Z}_{2k}^0 = \mathbb{Q}\zeta^k(2)$ and $\mathcal{Z}_{2k+1}^0 = \{0\}$.[1] There are plenty of relations between MZV. The following well-known result, conjectured in [6], has been proven analytically [34], via double-shuffle relations [12,21,28] and from associator relations [22].

Theorem 1.1 *(Parity for MZV). Whenever the weight w and depth d are of opposite parity, then $\mathcal{Z}_w^d = \mathcal{Z}_w^{d-1}$. In other words, $\zeta(n_1, \ldots, n_d)$ is a $\mathbb{Q}[\zeta(2)]$-linear combination of MZV of depth at most $d-1$, provided that $|\boldsymbol{n}| + d$ is odd.*

This theorem implies $\zeta(2k) \in \mathbb{Q}\zeta^k(2)$ in depth one; $\zeta(1,2) \in \mathbb{Q}\zeta(3)$ and $\zeta(2,3) \in \mathbb{Q}\zeta(5) + \mathbb{Q}\zeta(2)\zeta(3)$ are examples in depth two and an explicit witness for a reduction from depth 3 is (taken from [1])

$$\zeta(1,5,2) = \tfrac{703}{875}\zeta^4(2) - \tfrac{17}{2}\zeta(3)\zeta(5) - \tfrac{7}{10}\zeta(3,5) + 2\zeta(2)\zeta^2(3). \tag{1.3}$$

Note that there are two products on MZV, known as shuffle and quasi-shuffle (also called stuffle), which express a product $\zeta(\boldsymbol{n})\zeta(\boldsymbol{m})$ as a linear combination of MZV with integer coefficients [21]. For example, $\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$ shows that the right-hand side of (1.3) is indeed in $\mathcal{Z}_8^2$. In general, the products ensure that $\mathcal{Z}_w^d \cdot \mathcal{Z}_{w'}^{d'} \subseteq \mathcal{Z}_{w+w'}^{d+d'}$.

Thinking of MZV as special values $\zeta(\boldsymbol{n}) = \mathrm{Li}_{\boldsymbol{n}}(1, \ldots, 1)$ of multiple polylogarithms (MPL), defined by the series [17]

$$\mathrm{Li}_{\boldsymbol{n}}(\boldsymbol{z}) = \mathrm{Li}_{n_1,\ldots,n_d}(z_1, \ldots, z_d) := \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1} \cdots z_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}}, \tag{1.4}$$

raises the question if Theorem 1.1 also applies for other values of $\boldsymbol{z}$. The case when all $z_i \in \mu_N := \left\{ z \in \mathbb{C} \colon \ z^N = 1 \right\}$ are N-th roots of unity has been of particular interest [18,38], partly because such numbers occur in particle physics [7–9]. We set $\mathrm{Li}_\emptyset := 1$ in weight zero and write

$$\mathcal{Z}_w^d(\mu_N) := \mathrm{lin}_{\mathbb{Q}} \left\{ (2\pi i)^k \,\mathrm{Li}_{\boldsymbol{n}}(\boldsymbol{z}) \colon \ \boldsymbol{n} \in \mathbb{N}^r, \boldsymbol{z} \in \mu_N^r, |\boldsymbol{n}| + k = w, r \leq d \right\} \tag{1.5}$$

where, in contrast to (1.2), $n_r = 1$ is allowed as long as $z_r \neq 1$ (this ensures convergence) and k is restricted to even values in the cases $N = 1, 2$.

[1] This definition is natural to our approach via polylogarithms (powers of π can be generated from $\log(z)$, which has depth zero). It also simplifies Theorem 1.1 in abolishing the need to state it modulo products.