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Trace map and regularity of finite extensions of a DVR

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ABSTRACT

We interpret the regularity of a finite and flat extension of a discrete valuation ring in terms of the trace map of the extension.

1. Introduction

Let R be a ring and A be an R-algebra which is projective and finitely generated as R-module. We denote by $\mathrm{tr}_{A/R}\colon A \longrightarrow R$ the trace map and by $\widetilde{\mathrm{tr}}_{A/R}\colon A \longrightarrow A^\vee = \mathrm{Hom}_R(A, R)$ the map $a \longmapsto \mathrm{tr}_{A/R}(a \cdot -)$. It is a well known result of commutative algebra that the étaleness of the extension A/R is entirely encoded in the trace map $\mathrm{tr}_{A/R}$: A/R is étale if and only if the map $\widetilde{\mathrm{tr}}_{A/R}\colon A \longrightarrow A^\vee$ is an isomorphism (see [Gro71, Proposition 4.10]). In this case, if R is a DVR (discrete valuation ring) it follows

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that A is regular (that is a product of Dedekind domains), while the converse is clearly not true because extensions of Dedekind domains are often ramified.

In this paper we show how to read the regularity of A in terms of the trace map $\mathrm{tr}_{A/R}$. In order to express our result we need some notations and definitions. Let us assume from now on that the ring R is a DVR with residue field k_R. We first extend the notion of tame extensions and ramification index: given a maximal ideal p of A we set

$$e(p, A/R) = \frac{\dim_{k_R}(A_p \otimes_R k_R)}{[k(p) : k_R]}$$

where $k(p) = A/p$, and we call it the *ramification index* of p in the extension A/R. Notice that $e(p, A/R) \in \mathbb{N}$ (see Lemma 2.3). We say that A/R is *tame* (over the maximal ideal of R) if the ramification indexes of all maximal ideals of A are coprime with $\mathrm{char}\, k_R$. Those definitions agree with the usual ones when A is a Dedekind domain. We also set

$$\mathcal{Q}_{A/R} = \mathrm{Coker}(A \overset{\widetilde{\mathrm{tr}}_{A/R}}{\longrightarrow} A^\vee), \;\; f^{A/R} = \mathrm{l}(\mathcal{Q}_{A/R})$$

where l denotes the length function. Alternatively $f^{A/R}$ can be seen as the valuation of the discriminant section $\det \widetilde{\mathrm{tr}}_{A/R}$. We also denote by $|\operatorname{Spec}(A \otimes_R \overline{k_R})|$ the number of primes of $A \otimes_R \overline{k_R}$: this number can also be computed as

$$|\operatorname{Spec}(A \otimes_R \overline{k_R})| = \sum_{p \text{ maximal ideals of } A} [F_p : k_R]$$

where F_p denotes the maximal separable extension of k_R inside $k(p) = A/p$ (see Corollary 2.4). Finally we will say that A/R has separable residue fields (over the maximal ideal of R) if for all maximal ideals p of A the finite extension $k(p)/k_R$ is separable. The theorem we are going to prove is the following:

Main theorem. *Let R be a DVR and A be a finite and flat R-algebra. Then we have the inequality*

$$f^{A/R} \geq \mathrm{rk}\, A - |\operatorname{Spec}(A \otimes_R \overline{k_R})|$$

and the following conditions are equivalent:

1) *the equality holds in the inequality above;*
2) *A is regular and A/R is tame with separable residue fields;*
3) *A/R is tame with separable residue fields and the R-module $\mathcal{Q}_{A/R}$ is defined over k_R, that is $m_R\mathcal{Q}_{A/R} = 0$, where m_R denotes the maximal ideal of R.*