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Advanced refinements of Young and Heinz inequalities

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ABSTRACT

In this article, we prove several multi-term refinements of Young type inequalities for both real numbers and operators improving several known results. Among other results, we prove that for all $0 \le \nu \le 1$ and each $N \in \mathbb{N}$,

$$A\#_\nu B + \sum_{j=1}^{N} s_j(\nu)\left(A\#_{\alpha_j(\nu)}B + A\#_{2^{1-j}+\alpha_j(\nu)}B - 2A\#_{2^{-j}+\alpha_j(\nu)}B\right) \le A\nabla_\nu B,$$

for the positive operators A and B, where $s_j(\nu)$ and $\alpha_j(\nu)$ are certain functions. Moreover, some new Heinz type inequalities involving the Hilbert–Schmidt norm are established.

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1. Introduction

The simple inequality

$$a^{\nu}b^{1-\nu} \leq \nu a + (1-\nu)b, a, b > 0, 0 \leq \nu \leq 1 \tag{1.1}$$

is the celebrated Young inequality. Even though this inequality looks very simple, it is of great interest in operator theory. We refer the reader to [5,6,3,9,10,14] as a sample of the extensive use of this inequality in this field. Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

Among the first refinements of this inequality is the squared version proved in [3]

$$(a^{\nu}b^{1-\nu})^2 + \min\{\nu, 1-\nu\}^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2. \tag{1.2}$$

Later, the authors in [5] obtained the other interesting refinement

$$a^{\nu}b^{1-\nu} + \min\{\nu, 1-\nu\}(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b. \tag{1.3}$$

A common fact about the above refinements is having one refining term.

In the recent paper [13], some reverses and refinements of Young's inequality were presented, with one or two refining terms. In particular, it is proved that

$$\begin{cases} a^{\nu}b^{1-\nu} + \nu(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, & 0 < \nu \leq \frac{1}{2} \\ a^{\nu}b^{1-\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{a})^2 \leq \nu a + (1-\nu)b, & \frac{1}{2} < \nu \leq 1 \end{cases}, \tag{1.4}$$

where $r_0 = \min\{2r, 1-2r\}$ for $r = \min\{\nu, 1-\nu\}$. These inequalities refine (1.3) by adding a second refining term to the original Young's inequality. In the same paper, the following reversed versions have been proved too.

$$\begin{cases} \nu a + (1-\nu)b + r_0(\sqrt[4]{ab} - \sqrt{a})^2 \leq a^{\nu}b^{1-\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2, & 0 \leq \nu \leq \frac{1}{2} \\ \nu a + (1-\nu)b + r_0(\sqrt[4]{ab} - \sqrt{b})^2 \leq a^{\nu}b^{1-\nu} + \nu(\sqrt{a} - \sqrt{b})^2, & 0 \leq \nu \leq \frac{1}{2} \end{cases} \tag{1.5}$$

where $r_0 = \min\{2r, 1-2r\}$ for $r = \min\{\nu, 1-\nu\}$. These inequalities refine the reversed version $\nu a + (1-\nu)b \leq a^{\nu}b^{1-\nu} + \max\{\nu, 1-\nu\}(\sqrt{a} - \sqrt{b})^2$; cf. [6].

Moreover, it is proved in [13] that

$$\begin{cases} (\nu a + (1-\nu)b)^2 + r_0(\sqrt{ab} - a)^2 \leq \left(a^{\nu}b^{1-\nu}\right)^2 + (1-\nu)^2(a-b)^2, & 0 \leq \nu \leq \frac{1}{2} \\ (\nu a + (1-\nu)b)^2 + r_0(\sqrt{ab} - b)^2 \leq \left(a^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2, & 0 \leq \nu \leq \frac{1}{2} \end{cases} \tag{1.6}$$

refining the squared version $(\nu a + (1-\nu)b)^2 \leq \left(a^{\nu}b^{1-\nu}\right)^2 + \max\{\nu, 1-\nu\}^2(a-b)^2$ of [6].