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A consequence of Greenberg's generalized conjecture on Iwasawa invariants of $\mathbb{Z}_p$-extensions

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A B S T R A C T

For a prime number p and a number field k, let $\tilde{k}$ be the compositum of all $\mathbb{Z}_p$-extensions of k. Greenberg's Generalized Conjecture (GGC) claims the pseudo-nullity of the unramified Iwasawa module $X(\tilde{k})$ of $\tilde{k}$. It is known that, when k is an imaginary quadratic field, GGC has a consequence on the Iwasawa invariants associated to $\mathbb{Z}_p$-extensions of k. In this paper, we partially generalize it to arbitrary number fields k.

1. Introduction

Let p be a fixed prime number. We fix an algebraic closure of the field $\mathbb{Q}$ of rational numbers and any algebraic extension of $\mathbb{Q}$ is considered to be contained in it.

First we introduce some general notions in Iwasawa theory. For any algebraic extension F of $\mathbb{Q}$, let $L(F)$ be the maximal unramified pro-p abelian extension of F and let $X(F)$ be the Galois group $\mathrm{Gal}(L(F)/F)$. When k is a number field (i.e. a finite extension of $\mathbb{Q}$), it is known by class field theory that $X(k)$ is canonically isomorphic to the p-Sylow subgroup of the ideal class group of k. The structure of $X(F)$ is one of the main objects of study in number theory.

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Let k be a number field and d a positive integer. When K/k is a $\mathbb{Z}_p^d$-extension, let $\Lambda(K/k)$ be the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$, which is often called the Iwasawa algebra. It is known that $\Lambda(K/k)$ is non-canonically isomorphic to the ring of formal power series $\mathbb{Z}_p[[T_1, \dots, T_d]]$ and, in particular, $\Lambda(K/k)$ is a regular local ring. In fact, if $\sigma_1, \dots, \sigma_d$ constitute a $\mathbb{Z}_p$-basis of $\mathrm{Gal}(K/k)$, then an isomorphism $\Lambda(K/k) \xrightarrow{\sim} \mathbb{Z}_p[[T_1, \dots, T_d]]$ is obtained by sending σ_i to $1 + T_i$. Since $L(K)/k$ is a Galois extension, we have the natural action of $\mathrm{Gal}(K/k)$ on $X(K)$ via the inner automorphisms. This action defines the natural $\Lambda(K/k)$-module structure on $X(K)$. It is known that $X(K)$ is a finitely generated torsion $\Lambda(K/k)$-module. (See [4]. Although the statement there is the case where $K = \tilde{k}$ defined below, one can modify the proof to arbitrary multiple $\mathbb{Z}_p$-extensions.)

In particular, for any number field k, let $\tilde{k}$ be the compositum of all $\mathbb{Z}_p$-extensions of k. It is known that $\tilde{k}/k$ is a $\mathbb{Z}_p^{r_2(k)+1+\delta(k,p)}$-extension, where $r_2(k)$ is the number of complex places of k and $\delta(k,p)$ is the Leopoldt's defect of (k,p) (see [14, Proposition (10.3.20)]). We put $d(k) = r_2(k) + 1 + \delta(k,p)$, so $\tilde{k}/k$ is a $\mathbb{Z}_p^{d(k)}$-extension.

We also need some ring theoretic materials [14, Chapter V, §1]. In general, let Λ be a noetherian integrally closed domain and X a Λ-module. We say that X is a *pseudo-null* Λ-module and write $X \sim 0$ or more precisely $X \sim_\Lambda 0$ if X is finitely generated and the height of the annihilator ideal of X is greater than or equal to 2. A homomorphism $X \to Y$ of Λ-modules is said to be a pseudo-isomorphism if its kernel and cokernel are both pseudo-null. If there exists a pseudo-isomorphism $X \to Y$, we write $X \sim Y$ or $X \sim_\Lambda Y$.

Now Greenberg's Generalized Conjecture (GGC) claims the following.

Conjecture 1.1 *([6, Conjecture 3.5]). For any number field k, $X(\tilde{k})$ is pseudo-null as a $\Lambda(\tilde{k}/k)$-module.*

We say that GGC holds for (k,p) if $X(\tilde{k})$ is pseudo-null as a $\Lambda(\tilde{k}/k)$-module. Although GGC is still an open problem, there are some cases where GGC is known to be true. For example, GGC holds for (k,p) if k is an imaginary quadratic field and p does not divide the class number of k ([13, Proposition 3.A]). Moreover, there is a sufficient condition for GGC to hold in the case where k is a CM-field and p splits completely in $k/\mathbb{Q}$ ([2, Theorem 2]).

In this paper we focus on some consequences of GGC on the size of $X(K)$ for (multiple) $\mathbb{Z}_p$-extensions K of k. To state the main result, recall the definitions of the Iwasawa λ, μ, ν-invariants of a $\mathbb{Z}_p$-extension K/k. Let k_n be the n-th layer of K/k, in other words, the intermediate field of K/k such that $\mathrm{Gal}(K/k_n) = \mathrm{Gal}(K/k)^{p^n}$. Then there are unique non-negative integers $\lambda(K/k), \mu(K/k)$ and an integer $\nu(K/k)$ such that

$$\sharp X(k_n) = p^{\lambda(K/k)n + \mu(K/k)p^n + \nu(K/k)}$$

for sufficiently large n (see [16, Theorem 13.13]). In the case where k is an imaginary quadratic field, the following theorem is known.