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Wallis' sequence estimated accurately using an alternating series

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ABSTRACT

An asymptotic approximation of Wallis' sequence $m \mapsto W_m := \prod_{k=1}^{m} \frac{4k^2}{4k^2-1}$ is presented as

$$W_m = \frac{m\pi}{2m+1} \exp\left(2\sigma_q(m)\right) \cdot \exp\left(r_q(m)\right),$$

where

$$\sigma_q(x) := \sum_{i=1}^{\lfloor q/2 \rfloor} \frac{\left(1-4^{-i}\right) B_{2i}}{i(2i-1) \cdot x^{2i-1}}$$

(B_k are the Bernoulli coefficients),

and where

$$|r_q(m)| < r_q^*(m) := \frac{2\pi(q-2)!}{3(2m\pi)^{q-1}},$$

for any integers $m \geq 1$ and $q \geq 2$.

Parameters m and q control the error factor $\exp\left(r_q(m)\right)$.

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1. Introduction

The Wallis sequence $(W_n)_{n\geq 1}$ defined as

$$W_n := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \tag{1}$$

is clearly strictly increasing and was used by English mathematician Wallis[1] in 1655 [15,16] to introduce π as a limit:

$$\frac{\pi}{2} = \lim_{n\to\infty} W_n. \tag{2}$$

This is in the history the first presentation of π as a limit of an analytically given sequence. Wallis' sequence was investigated by many authors since it is closely related to the constant π, see for example [2–5,15]. Although Wallis' sequence was usually considered as unsuitable for numerical computation of π, it was shown in [8] and [13] that it is usable also for computation of some decimals of π. Moreover, knowing the value of π, it is possible to obtain rather good approximations of W_n. But W_n is closely related with Catalan numbers $c_n := \frac{1}{n+1}\binom{2n}{n}$ which play important role in combinatorics and the theory of graphs, see e.g. [7]. The connection is given through the formula

$$c_n = \frac{4^n}{(n+1)\sqrt{2n+1}} \cdot \frac{1}{\sqrt{W_n}} \qquad (n \in \mathbb{N}).$$

All these and similar facts have attracted mathematicians to study Wallis' sequence for a very long period of time. Consequently, during the time a great amount of articles about Wallis' sequence have been published, recently [6,8,11,13,14].

In [6] are given the following three main results:

[6, Theorem 1] For all $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right) < W_n < \frac{\pi}{2}\left(1-\frac{1}{4n+\beta}\right)$$

with the best possible constants $\alpha = 5/2$ and $\beta = 2.614\ldots$.

[6, Theorem 2] For all $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+5/2}\right)^{\lambda} < W_n < \frac{\pi}{2}\left(1-\frac{1}{4n+5/2}\right)^{\mu}$$

with the best possible constants $\lambda = 1$ and $\mu = 0.981\ldots$.

[1] John Wallis, 1616–1703.