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The p-adic representation of the Weil–Deligne group associated to an abelian variety

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ABSTRACT

Let A be an abelian variety defined over a number field $F \subset \mathbf{C}$ and let G_A be the Mumford–Tate group of $A_{/\mathbf{C}}$. After replacing F by a finite extension, we can assume that, for every prime number ℓ, the action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}}, \mathbf{Q}_\ell)$ factors through a map $\rho_\ell \colon \Gamma_F \to G_A(\mathbf{Q}_\ell)$.
Fix a valuation v of F and let p be the residue characteristic at v. For any prime number $\ell \neq p$, the representation ρ_ℓ gives rise to a representation $'W_{F_v} \to G_{A/\mathbf{Q}_\ell}$ of the Weil–Deligne group. In the case where A has semistable reduction at v it was shown in a previous paper that, with some restrictions, these representations form a compatible system of $\mathbf{Q}$-rational representations with values in G_A.
The p-adic representation ρ_p defines a representation of the Weil–Deligne group $'W_{F_v} \to G^\iota_{A/F_{v,0}}$, where $F_{v,0}$ is the maximal unramified extension of $\mathbf{Q}_p$ contained in F_v and G^ι_A is an inner form of G_A over $F_{v,0}$. It is proved, under the same conditions as in the previous theorem, that, as a representation with values in G_A, this representation is $\mathbf{Q}$-rational and that it is compatible with the above system of representations $'W_{F_v} \to G_{A/\mathbf{Q}_\ell}$.

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0. Introduction

This paper is dedicated to the comparison of the étale and the log-crystalline cohomologies of an abelian variety over a number field. More precisely, it treats the way in which the action of the Galois group of the base field on the étale cohomology is reflected in the crystalline theory. We start with a succinct overview of relevant conjectures and results, referring to the introduction of [Noo13] for a more detailed discussion concerning the system of representations afforded by the étale cohomology.

Let X be a proper and smooth variety over a finite extension F_v of $\mathbf{Q}_p$. For any prime number ℓ and any i, the absolute Galois group $\Gamma_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$ acts on the étale cohomology group $\mathrm{H}^i_{\text{ét}}(X_{/\bar{F}_v}, \mathbf{Q}_\ell)$. If $\ell \neq p$, the corresponding representation of Γ_{F_v} gives rise to an ℓ-adic representation of the Weil–Deligne group $'W_{F_v}$ of F_v. For general X, it is conjectured that these representations are $\mathbf{Q}$-rational and that, for fixed i and variable $\ell \neq p$, they form a compatible system of representations of $'W_{F_v}$, see [Del73] and [Fon94b, 2.4]. In the case where X has good reduction, the étale cohomology of $X_{\bar{F}_v}$ is isomorpic to the étale cohomology of the special fibre of a proper and smooth model of X over the valuation ring of F_v and hence the inertia subgroup of Γ_{F_v} acts trivially. In this case, the conjecture comes down to the fact that the characteristic polynomial of the Frobenius element has rational coefficients and that it is independent of ℓ. This is a consequence of the Weil conjectures proved by Deligne.

Now assume that $X = A$ is an abelian variety, not necessarily with good reduction. As $\mathrm{H}^i_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell) \cong \wedge^i \mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ for every i, it is harmless to assume that $i = 1$. In this case, it is well known that the above conjecture is true, cf. [Del73, Exemple 8.10]. There is a more precise result in the case where A can be defined over a number field $F \subset F_v$ for which we also fix an embedding $F \subset \mathbf{C}$. In this case, the Mumford–Tate group G_A of A is defined. It is a linear algebraic group and $G_{A/\mathbf{Q}_\ell}$ acts on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ for every ℓ. Up to a finite extension of F, the representation of Γ_{F_v} on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ factors through $G_A(\mathbf{Q}_\ell)$ for all ℓ. For $\ell \neq p$, it follows that the associated representation of $'W_{F_v}$ factors through $G_{A/\mathbf{Q}_\ell}$. Under the hypothesis that A has semistable reduction and with a number of other restrictions, it is shown in [Noo13] that these representations of $'W_{F_v}$ with values in $G_{A/\mathbf{Q}_\ell}$ are $\mathbf{Q}$-rational and that they are pairwise conjugate for the action of a group containing G_A with finite index. The precise statement is recalled in Theorem 3.3. Note that the fact that A has semistable reduction at v implies that the inertia group $I_{F_v} \subset W_{F_v} \subset {}'W_{F_v}$ acts trivially on $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}_v}, \mathbf{Q}_\ell)$ so that the representation of $'W_{F_v}$ is determined by the action of the monodromy operator N and the image the Frobenius element. The case of an abelian variety with good reduction, for which the monodromy N is also trivial, had previously been treated in [Noo09]. A more general motivic conjecture is proposed by Serre [Ser94, §12].

The main Theorem 3.8 of this paper extends the compatibility result for the ℓ-adic representations to the p-adic representation. In order to get an idea of the statement, it is useful to return briefly to the general case of a variety X/F_v with good reduction. Contrary to the its action on the ℓ-adic cohomology, the action of Γ_{F_v} on the