Journal of Number Theory 172 (2017) 321–342

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Davenport constant for commutative rings

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ARTICLE INFO

Article history:
Received 11 February 2016
Received in revised form 20 May 2016
Accepted 23 August 2016
Available online 11 October 2016
Communicated by D. Goss

Keywords:
Davenport constant
Semigroup
Finite rings

ABSTRACT

The Davenport constant is one measure for how "large" a finite abelian group is. In particular, the Davenport constant of an abelian group is the smallest k such that any sequence of length k is reducible. This definition extends naturally to commutative semigroups, and has been studied in certain finite commutative rings. In this paper, we give an exact formula for the Davenport constant of a general commutative ring in terms of its unit group.

1. Introduction

The Davenport constant is an important concept in additive number theory. In particular, it measures the largest zero-free sequence of an abelian group.

The Davenport constant was introduced by Davenport in 1966 [3], but was actually studied prior to that in 1963 by Rogers [8]. The definition was first extended by Geroldinger and Schneider to abelian semigroups in [4] as follows:

Definition 1.1. For an additive abelian semigroup S, let $d(S)$ denote the smallest $d \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

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http://dx.doi.org/10.1016/j.jnt.2016.08.001
0022-314X/

For any $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ there exists a subset $J \subseteq [1, m]$ such that $|J| \leq d$ and

$$\sum_{j=1}^{m} s_j = \sum_{j \in J} s_j.$$

In addition, [4] showed the following:

Proposition 1.2. *If $|S| < \infty$, then $d(S) < \infty$.*

Wang and Gao [11] then gave the definition of the large Davenport constant in terms of reducible and irreducible sequences, as follows:

Definition 1.3. Let S be a commutative semigroup (not necessarily finite). Let A be a sequence of elements in S. We say that A is *reducible* if there exists a proper subsequence $B \subsetneq A$ such that the sum of the elements in A is equal to the sum of the elements in B. Otherwise, we say that A is *irreducible*.

Definition 1.4. Let S be a finite commutative semigroup. Define the *Davenport constant* $D(S)$ of S as the smallest $d \in \mathbb{N} \cup \{\infty\}$ such that every sequence S of d elements in S is reducible.

Remark. The quantities D and d are related by the equation $D(S) = d(S) + 1$.

Note that if S is an abelian group, being irreducible is equivalent to being zero-sum free, so the definition of the Davenport constant here is equivalent to the classical definition of the Davenport constant for abelian groups.

In all following sections, unless otherwise noted:

- All semigroups are unital and commutative. Furthermore, we will use multiplication notation for semigroups, as opposed to the additive convention used in [4,11,10,13].
- Similarly, rings are unital and commutative.
- Sets represented by the capital letter S are semigroups.
- Sets represented by the capital letter T are ideals in a semigroup.
- Sets represented by the capital letters A, B are sequences of elements in a semigroup. In addition, $\pi(A)$ denotes the product of the elements in A.
- Sets represented by the capital letter R are commutative rings.
- By abuse of notation, when we write $D(R)$, we actually mean $D(S_R)$, where S_R is the semigroup of R *under multiplication*.
- C_n denotes the cyclic group of order n; $\mathbb{Z}/n\mathbb{Z}$ denotes the ring with additive group C_n.

In Section 2 we review previous results about the Davenport constant as it relates to the multiplicative semigroup of specific commutative rings. In Section 3 we present