Journal of Number Theory 172 (2017) 343–382

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Partial sums of biased random multiplicative functions

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A R T I C L E I N F O

Article history:
Received 21 July 2015
Received in revised form 12 August 2016
Accepted 15 August 2016
Available online 21 October 2016
Communicated by K. Soundararajan

Keywords:
Random multiplicative functions
Probabilistic Number Theory
Riemann Hypothesis

A B S T R A C T

Let $\mathcal{P}$ be the set of the primes. We consider a class of random multiplicative functions f supported on the squarefree integers, such that $\{f(p)\}_{p\in\mathcal{P}}$ form a sequence of ± 1 valued independent random variables with $\mathbb{E}f(p) < 0$, $\forall p \in \mathcal{P}$. The function f is called strongly biased (towards classical Möbius function), if $\sum_{p\in\mathcal{P}} \frac{f(p)}{p} = -\infty$ *a.s.*, and it is weakly biased if $\sum_{p\in\mathcal{P}} \frac{f(p)}{p}$ converges *a.s.* Let $M_f(x) := \sum_{n\leq x} f(n)$. We establish a number of necessary and sufficient conditions for $M_f(x) = o(x^{1-\alpha})$ for some $\alpha > 0$, *a.s.*, when f is strongly or weakly biased, and prove that the Riemann Hypothesis holds if and only if $M_{f_\alpha}(x) = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$ *a.s.*, for each $\alpha > 0$, where $\{f_\alpha\}_\alpha$ is a certain family of weakly biased random multiplicative functions.

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http://dx.doi.org/10.1016/j.jnt.2016.08.020

1. Introduction

A function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative function if $f(1) = 1$ and $f(nm) = f(n)f(m)$ whenever n and m are coprime. Let $\mathcal{P}$ be the set of the prime numbers. In this paper we consider a class of multiplicative functions f which are supported on the square-free integers, *i.e.* $f(n) = 0$ for all $n \in \mathbb{N}$, for which $\exists\, p \in \mathcal{P}$ such that $p^2|n$. A function f from this class is called random (binary) multiplicative function if $\{f(p)\}_{p\in\mathcal{P}}$ form a sequence of ± 1 valued independent random variables.

Let μ be the Möbius function, the multiplicative function supported on the square-free integers with $\mu(p) = -1\, \forall p \in \mathcal{P}$. We say that f is *biased* (towards μ) if $\mathbb{E}f(p) < 0\, \forall p \in \mathcal{P}$. If f is biased and $\sum_{p\in\mathcal{P}} \frac{f(p)}{p}$ converges *a.s.*, we say that f is *weakly biased*; otherwise, if $\sum_{p\in\mathcal{P}} \frac{f(p)}{p} = -\infty$ *a.s.*, we say that f is *strongly biased*. In the case $(f(p))_{p\in\mathcal{P}}$ is i.i.d. with $\mathbb{E}f(2) = 0$, we say that f is an unbiased random multiplicative function.

Further, for $x \geq 1$, we denote $M_f(x) := \sum_{n\leq x} f(n)$.

A classical result of J.E. Littlewood, [19], states that the Riemann Hypothesis (RH) holds if and only if the Mertens' function $M_\mu(x) = o(x^{1/2+\epsilon})$, $\forall \epsilon > 0$. This criterion led A. Wintner to investigate what happens with the partial sums $M_f(x)$ of a random multiplicative function f. In [27], A. Wintner proved that if f is unbiased, then $M_f(x) = o(x^{1/2+\epsilon})$, $\forall \epsilon > 0$ *a.s.* Since then, many results pursuing the exact order of $M_f(x)$ have been proved [10,13,2,14,18], and also Central Limit Theorems have been established [16,15,7].

These results naturally raise a question of what can be said for $M_f(x)$ in the case of biased f. Since we have A. Wintner's Theorem in the case that f is unbiased, by the probabilistic line of reasoning, a natural prediction is that, if f has a sufficiently small bias, then $M_f(x) = o(x^{1/2+\epsilon})$, $\forall \epsilon > 0$ *a.s.* Further, by the same probabilistic point of view and Littlewood's criterion, one can expect to reformulate RH in terms of $M_f(x)$ in the case that f has a strong bias.

To illustrate this reasoning, consider a strongly biased f such that the series $\sum_{p\in\mathcal{P}}(1+\mathbb{E}f(p))$ converges. In this case, we informally say that f is essentially μ, since by the Borel–Cantelli Lemma, the random subset of primes $\{p \in \mathcal{P} : f(p) \neq \mu(p)\}$ is finite *a.s.* In particular, $M_\mu(x)$ and $M_f(x)$ have essentially the same asymptotic behavior (see Lemma A.3 and Lemma A.4). On the other hand, a similar argument shows that if $\sum_{p\in\mathcal{P}} \mathbb{E}f(p)$ converges, then f is essentially unbiased and hence, as a consequence from Wintner's Theorem, $M_f(x) = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$ *a.s.*

In this paper we are interested in determining the range of biased f such that $M_f(x) = o(x^{1/2+\epsilon})$ $\forall \epsilon > 0$ *a.s.*, and the range of biased f such that we can reformulate RH in terms of the asymptotic behavior of $M_f(x)$. In the case that f has a small bias, our first result states:

Theorem 1.1. *Let $\alpha > 0$ and f_α is such that $\mathbb{E}f_\alpha(p) = -\frac{1}{p^\alpha}\, \forall p \in \mathcal{P}$. Then the Riemann hypothesis holds if and only if $M_{f_\alpha}(x) = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$ a.s., for each $\alpha > 0$.*