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On the field of definition of a cubic rational function and its critical points

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A B S T R A C T

Using essentially only algebra, we give a proof that a cubic rational function over $\mathbb{C}$ with real critical points is equivalent to a real rational function. We also show that the natural generalization to $\mathbb{Q}_p$ fails unless $p = 3$.

1. Introduction

Let K be a field of characteristic zero with algebraic closure $\bar{K}$. We say that two rational functions $f, g \in \bar{K}(z)$ are **equivalent** if there is a fractional linear transformation $\sigma \in \bar{K}(z)$ such that $f = \sigma \circ g$. Viewing f and g as endomorphisms of the projective line, we see that they are equivalent if they differ by a change of coordinate on the target. Note that equivalent rational functions have the same critical points. This is the equivalence

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relation used in the study of *dessins d'enfants*, as opposed to the equivalence used for dynamical systems.

Theorem 1.1 *(Eremenko/Gabrielov). If $f \in \mathbb{C}(z)$ is a rational function with real critical points, then f is equivalent to a rational function with real coefficients.*

By relating equivalence classes of rational functions with special Schubert cycles, Goldberg [3] showed that there are at most

$$\rho(d) := \frac{1}{d}\binom{2d-2}{d-1}$$

equivalence classes of degree d rational functions with a given set of critical points. Eremenko and Gabrielov [1,2] used topological, combinatorial, and complex analytic techniques to *construct* exactly $\rho(d)$ real rational functions with a given set of real critical points, which proves the theorem.

But the correspondence between a rational function and its critical points is purely algebraic, via roots of the derivative. This raises the question of whether a truly elementary proof of the above result exists—one that does not use any analysis or topology. We give such a proof for cubic functions in this note.

For a field K and a nonconstant rational function $\phi \in K(z)$, we say that K is **ϕ-perfect** if the map $\phi\colon \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is surjective. For example, if K has characteristic p and $\phi(z) = z^p$, then K is ϕ-perfect if and only if it is a perfect field in the usual sense.

Theorem 1.2. *Let K be a field of characteristic zero with algebraic closure $\bar{K}$. The following statements are equivalent:*

(1) *Any cubic rational function $f \in \bar{K}(z)$ with K-rational critical points is equivalent to a rational function in $K(z)$.*
(2) *K is ϕ-perfect for the function*

$$\phi(z) = -\frac{z^2+2z}{2z+3}.$$

(3) *$(2y-1)^2+3$ is a square in K for every $y \in K$.*

Theorem 1.2 will be proved in § 2.

Corollary 1.3. *If $f \in \mathbb{C}(z)$ is a cubic rational function with real critical points, then f is equivalent to a real rational function.*

Proof. Evidently $(2y-1)^2+3$ is a square in $\mathbb{R}$ for every $y \in \mathbb{R}$. □