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# On partitions with fixed number of even-indexed and odd-indexed odd parts

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A B S T R A C T

This article is an extensive study of partitions with fixed number of odd and even-indexed odd parts. We use these partitions to generalize recent results of C. Savage and A. Sills. Moreover, we derive explicit formulas for generating functions for partitions with bounds on the largest part, the number of parts and with a fixed value of BG-rank or with a fixed value of alternating sum of parts. We extend the work of C. Boulet, and as a result, obtain a four-variable generalization of Gaussian binomial coefficients. In addition, we provide combinatorial interpretation of the Berkovich–Warnaar identity for Rogers–Szegő polynomials.



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## 1. Introduction and notation

A partition $\pi$ is a non-increasing finite sequence $\pi = (\lambda_1, \lambda_2, \dots)$ of positive integers. The elements $\lambda_i$ that appear in the sequence $\pi$ are called parts of $\pi$. For positive integers $i$, we call $\lambda_{2i-1}$ odd-indexed parts, and $\lambda_{2i}$ even indexed parts of $\pi$. We say $\pi$ is a partition of $n$, if the sum of all parts of $\pi$ is equal to $n$. Conventionally the empty sequence is considered as the unique partition of zero. We will abide by this convention. Partitions can be represented graphically in multiple ways. For consistency, we are going to focus on representing partitions using Ferrers diagrams. The 2-residue Ferrers diagram of partition $\pi$ is given by taking the ordinary Ferrers diagram drawn with boxes instead of dots and filling these boxes using alternating 0's and 1's starting from 0 on odd-indexed parts and 1 on even-indexed parts. We can exemplify 2-residue diagrams with $\pi = (12, 10, 7, 5, 2)$ in Table 1.

**Table 1**
2-residue Ferrers diagram of the partition $\pi = (12, 10, 7, 5, 2)$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | | | | | |
| 1 | 0 | 1 | 0 | 1 | | | | | | | |
| 0 | 1 | | | | | | | | | | |

In this paper, we consider partitions with fixed number of odd and even-indexed odd parts. We start our discussion by considering partitions into distinct parts. In this way we are led to Theorem 1.1.

**Theorem 1.1.** *For non-negative integers $i$, $j$, and $n$*

$$p(i, j, n) = p'(i, j, n),$$

*where $p(i, j, n)$ is the number of partitions of $n$ into distinct parts with $i$ odd-indexed odd parts and $j$ even-indexed odd parts and $p'(i, j, n)$ is the number of partitions of $n$ into distinct parts with $i$ parts that are congruent to 1 modulo 4, and $j$ parts that are congruent to 3 modulo 4.*

We can demonstrate Theorem 1.1 for special choices $i = j = 1$, and $n = 14$ in Table 2.

Let $P(i, j, q)$ be the generating function for $p(i, j, n)$:

$$P(i, j, q) = \sum_{n\geq 0} p(i, j, n)q^n.$$

Let $L$, $k$, $n$, $m$ be non-negative integers. We will use standard notations in [3] and [12].