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# Are number fields determined by Artin $L$-functions?

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ABSTRACT

Let $k$ be a number field, $K/k$ a finite Galois extension with Galois group $G$, $\chi$ a faithful character of $G$. We prove that the Artin $L$-function $L(s, \chi, K/k)$ determines the Galois closure of $K$ over $\mathbb{Q}$. In the special case $k = \mathbb{Q}$ it also determines the character $\chi$.



## 1. Introduction

Let $k$ be a number field, $K/k$ a finite Galois extension with Galois group $G$, $\chi$ a faithful character of $G$. In Theorem 6 we prove that the Artin $L$-function $L(s, \chi, K/k)$ determines the Galois closure $\tilde{K}$ of $K$ over $\mathbb{Q}$. In the special case $k = \mathbb{Q}$ we prove in Theorem 5 that the Artin $L$-function determines $K$ and the (faithful) character $\chi$. We

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give examples that in the case $k \neq \mathbb{Q}$ we cannot expect more, especially there exist non-isomorphic arithmetically equivalent fields which cannot be distinguished by Artin $L$-functions.

The restriction to faithful characters is natural: let $K/k$ be a finite normal extension with $\mathrm{Gal}(K/k) = G$, and let $\chi$ be a character of $G$ with $\mathrm{Ker}\,(\chi) \neq \{1\}$. Let $F$ be the fixed field of $\mathrm{Ker}\,(\chi)$, $H := G/\mathrm{Ker}\,(\chi)$ the Galois group of $F/k$, $\varphi : H \to \mathbb{C}$, $\varphi(\sigma\mathrm{Ker}\,(\chi)) := \chi(\sigma)$ for $\sigma \in G$. We have that

$$L(s, \chi, K/k) = L(s, \varphi, F/k),$$

and $\varphi$ is faithful.

As particular cases we obtain that the Dedekind zeta function of a number field determines its normal closure ([4], Theorem 1, p. 345) and that a Galois number field is determined by any Artin $L$-function corresponding to a character which contains all irreducible characters of the Galois group, the result of [3].

## 2. Properties of Artin $L$-functions

We do not give the definition of Artin $L$-functions, but we recall some fundamental properties of Artin $L$-functions needed in the sequel. Note that Artin $L$-functions are generalizations of Dedekind zeta functions $\zeta_K$ via

$$L(s, 1, K/K) = \zeta_K(s),$$

where $K$ is a number field and 1 is the trivial character of the trivial group $\mathrm{Gal}(K/K)$. We get further possibilities to write a Dedekind zeta function as an Artin $L$-function by using Propositions 1 and 2.

**Proposition 1.** *Let $k$ be a number field, $K/k$ a finite Galois extension with Galois group $G$, $\chi$ a character of $G$. Let $N$ be a finite Galois extension of $k$ which contains $K$, $U := \mathrm{Gal}(N/k)$, $V := \mathrm{Gal}(N/K)$. We identify the groups $G$ and $U/V$. Let*

$$\tilde{\chi} : U \to \mathbb{C},\ \tilde{\chi}(\sigma) := \chi(\sigma V).$$

*Then we have*

$$L(s, \tilde{\chi}, N/k) = L(s, \chi, K/k).$$

**Proof.** This follows straightforward from the definition of $L$-functions: [1], p. 297, formula (8). □

**Proposition 2.** *Let $k$ be a number field, $K/k$ a finite Galois extension with Galois group $G$. Let $k \subseteq F \subseteq K$ be an intermediate field, $H := \mathrm{Gal}(K/F)$, $\chi$ a character*