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The Banach–Mazur–Schmidt and Banach–Mazur–McMullen games

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ABSTRACT

We introduce two new mathematical games, the Banach–Mazur–Schmidt game and the Banach–Mazur–McMullen game, merging well-known games. We investigate the properties of the games, as well as providing an application to Diophantine approximation theory, analyzing the geometric structure of certain Diophantine sets.

1. Introduction

1.1. Schmidt's game and the Banach–Mazur game

The Banach–Mazur game, dating back to 1935, is arguably the prototype for all infinite mathematical games. This game has been extensively studied and we refer the

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interested reader to [13,11] for a thorough historical overview and recent developments. One of the most interesting aspects of the game is its connection to topology, namely that the second player has a winning strategy if and only if the target set is comeager.

In 1966, W. M. Schmidt [12] introduced a two-player game referred to thereafter as Schmidt's game. This game may be considered in a sense as a variant of the Banach–Mazur game. Schmidt invented the game primarily as a tool for studying certain sets which arise in number theory and Diophantine approximation theory. These sets are often exceptional with respect to both measure and category. The most significant example is the following. Let $\mathbb{Q}$ denote the set of rational numbers. A real number x is said to be *badly approximable* if there exists a positive constant $c = c(x)$ such that $\left|x - \frac{p}{q}\right| > \frac{c}{q^2}$ for all $\frac{p}{q} \in \mathbb{Q}$. We denote the set of badly approximable numbers by BA. This set plays a major role in Diophantine approximation theory, and is well-known to be both meager and Lebesgue null. Nonetheless, using his game, Schmidt was able to prove the following remarkable result:

Theorem 1.1 *(Schmidt [12]). Let $(f_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{C}^1$ diffeomorphisms of $\mathbb{R}$. Then the Hausdorff dimension of the set $\bigcap_{n=1}^{\infty} f_n^{-1}(\mathrm{BA})$ is 1. In particular, $\bigcap_{n=1}^{\infty} f_n^{-1}(\mathrm{BA})$ is uncountable.*

In this paper we shall combine these two games to form the Banach–Mazur–Schmidt game (or BMS game), as well as a variant called the Banach–Mazur–McMullen game, based on a variation of Schmidt's game introduced by C. T. McMullen [9]. These games are useful tools for studying certain questions arising in Diophantine approximation. In this paper, we use the BMS game to study the geometric properties of a certain Diophantine set (cf. Theorem 3.1 below), and in a recent preprint of the first- and third-named authors written jointly with R. Broderick [3], we used the BMS game to answer a question raised by Y. Bugeaud [4]. We strongly believe that many more applications will follow in the coming years. In addition, we show that the classes of sets which are winning for the BMS and BMM games can be compactly described by geometric conditions, in analogy with the characterization of Banach–Mazur winning sets as those which are comeager. This contrasts with Schmidt's game, for which no such characterization is known.

Remark 1.2. We shall describe the games in the context of complete metric spaces. One could consider a more general framework of topological games, but as all of our applications and results are in this more restricted context, we prefer not to follow the most general presentation.

1.2. Description of games

Let (X, d) be a complete metric space. In what follows, we denote by $B(x, r)$ and $B^{\circ}(x, r)$ the closed and open balls in the metric space (X, d) centered at x of radius r, i.e.,