Journal of Number Theory 167 (2016) 259–283

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Reductions of algebraic integers

Christophe Debry [a,b], Antonella Perucca [c,*]

[a] *Mathematics Department, KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium*
[b] *Korteweg–de Vries Institute for Mathematics, Universiteit van Amsterdam, Netherlands*
[c] *Fakultät Mathematik, Universität Regensburg, 93040 Regensburg, Germany*

ARTICLE INFO

Article history:
Received 29 September 2015
Received in revised form 8 March 2016
Accepted 8 March 2016
Available online 29 April 2016
Communicated by David Goss

MSC:
primary 11R44
secondary 11R18, 11R21, 11Y40

Keywords:
Number field
Algebraic integer
Reduction
Kummer theory
Density

ABSTRACT

Let K be a number field, and let G be a finitely generated subgroup of $K^\times$. Fix some prime number ℓ, and consider the set of primes $\mathfrak{p}$ of K satisfying the following property: the reduction of G modulo $\mathfrak{p}$ is well-defined and has size coprime to ℓ. We are able to give a closed-form expression for the natural density of this set.

* Corresponding author.
E-mail addresses: christophe.debry@wis.kuleuven.be (C. Debry), antonella.perucca@mathematik.uni-regensburg.de (A. Perucca).

http://dx.doi.org/10.1016/j.jnt.2016.03.001

1. Introduction

1.1. Motivation

Consider the number field $K = \mathbb{Q}(\zeta_8)$ and let G be respectively one of the following subgroups of $K^\times$:

$$\langle 12, 18\rangle \qquad \langle 12\cdot\zeta_8, 18\rangle \qquad \langle 12, 18\cdot\zeta_8\rangle\,.$$

The set of primes $\mathfrak{p}$ of K (not lying over 2 or 3) such that the reduction of G modulo $\mathfrak{p}$ has odd size admits a natural density. The densities for the given subgroups are $\frac{1}{56}$, $\frac{1}{56}$ and $\frac{1}{448}$ respectively. Why do we get precisely these densities, and what makes the third subgroup different from the first two? The aim of the paper is to understand these interesting phenomena.

For every number field K, for every finitely generated subgroup G of $K^\times$ and for every prime number ℓ we consider the set of primes $\mathfrak{p}$ of K satisfying the following property: *the reduction of G modulo $\mathfrak{p}$ is well-defined and has size coprime to ℓ*. This set admits a natural density and we prove a closed-form expression for this density, without imposing any additional conditions on the data. The paper is self-contained because it relies on classical results of algebraic number theory.

1.2. History

The problem that we solve in this paper has been open since the works of Hasse [2,3] in the 1960s, where the case of rank one for the field of rational numbers is settled. For an exhaustive survey about related questions (mostly for $\mathbb{Q}$ or for quadratic fields) we refer the reader to [7] by Moree. The second-named author [8] solved the case of rank one for every number field. Going in a different direction, Jones and Rouse [4] treated the case of rank one for commutative algebraic groups (in the generic case, i.e. assuming that the degrees of all relevant torsion fields and Kummer extensions are maximal). A complete solution to the problem for elliptic curves seems for the moment out of reach, however this provides a direction for future research.

1.3. The main result

Let K be a number field, and let G be a finitely generated and torsion-free subgroup of $K^\times$ of positive rank. Fix some prime number ℓ.

We always tacitly exclude the finitely many primes $\mathfrak{p}$ of K for which the reduction of G modulo $\mathfrak{p}$ is not a well-defined subgroup of $k_\mathfrak{p}^\times$ (the multiplicative group of the residue field at $\mathfrak{p}$). We want to compute and understand the natural density

$$D_{K,\ell}(G) := \operatorname{dens}\{\mathfrak{p} : \ell \nmid \sharp(G \bmod \mathfrak{p})\}\,.$$