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Curvature on the integers, II

Alexandru Buium

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA

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ABSTRACT

In a prequel to this paper [1] a notion of curvature on the integers was introduced, based on a formal patching technique. In this paper, which is essentially independent of its prequel, we introduce another notion of curvature on the integers, based on "algebraization of Frobenius lifts by correspondences." Our main results are vanishing/non-vanishing theorems for this new type of curvature in the case of "Chern connections" attached to classical groups.

1. Introduction

This paper is, in principle, a continuation of [1] but, from a logical standpoint, it is independent of [1]. For the motivation of our theory, and its comparison with classical differential geometry, we refer to the discussion in [1]. More generally, the present paper should be viewed as taking a step in the direction of developing a "differential geometry

E-mail address: buium@math.unm.edu.

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on *Spec* $\mathbb{Z}$"; this direction of research is consistent with the study in [4–6,1] of "arithmetic differential equations," as well as with Borger's viewpoint in [2] on the "field with one element."

In [1] we started by viewing the ring of integers, $\mathbb{Z}$, as an analogue of a ring of functions on an infinite dimensional manifold in which the various directions are given by the primes; then, in the spirit of [4,5,7], we replaced the partial derivative operators, acting on functions on a manifold, by Fermat quotient type operators, called *p-derivations*, acting on numbers. We then developed an arithmetic analogue of connections and curvature on the "manifold" *Spec* $\mathbb{Z}$ and we proved a series of vanishing/non-vanishing results for the curvature of "Chern connections" attached to the classical groups. In order to achieve this program we had to deal, in [1], with the following difficulty: the various p-derivations defining the Chern connections on GL_n are defined as self-maps of the corresponding p-adic completions of the ring of functions of GL_n so, when p varies, the p-derivations under consideration do not act on the same ring. Consequently, one cannot directly consider their commutator and, hence, their curvature. In [1] we overcame this problem by implementing a formal patching technique. The idea of formal patching goes back to work of Zariski who introduced it as a substitute for analytic continuation [14, Preface, pp. xii–xiii]. Formal patching also plays a prominent role in Galois theory [9]. The brand of formal patching used in [1] was introduced in [7,3] where it was referred to as *analytic continuation between primes*. This technique was helpful in [1] only in the case of classical groups defined by symmetric/antisymmetric matrices with entries roots of unity or zero. In the present paper we will overcome the above mentioned difficulty in a different way, namely by "algebraizing" the analytic picture in [1]. This algebraization method has at least two advantages:

1) it works for arbitrary symmetric/antisymmetric matrices, with entries not necessarily roots of unity or zero;

2) it deals with schemes, indeed with function fields of varieties, rather than with formal schemes.

The price to pay is that one needs to replace endomorphisms by correspondences. On the other hand correspondences can be composed and commutators can be attached to them, leading to a new concept of curvature. The resulting picture, in the present paper, will then acquire, as we shall see, a birational/motivic flavor. Our main results will, again, be vanishing/non-vanishing theorems for (this new type of) curvature in the case of "Chern connections" attached to the classical groups. Our *Chern connections* are analogues of the Chern connections on hermitian vector bundles [8, p. 73] and were introduced in [6, Introduction]; we will review their definition presently.

In the rest of this Introduction we give a rough idea of our main constructions and results. We begin by recalling from the Introduction to [1] a few basic definitions. Recall from [4,11] the following:

Definition 1.1. A *Frobenius lift* on a ring B is a ring endomorphism of B inducing the p-power Frobenius on B/pB.