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Congruences on the number of restricted m-ary partitions

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ABSTRACT

Andrews, Brietzke, Rødseth and Sellers proved an infinite family of congruences on the number of the restricted m-ary partitions when m is a prime. In this note, we show that these congruences hold for arbitrary positive integer m and thus confirm the conjecture of Andrews, et al.

1. Introduction

Let $m \geq 2$ be a fixed integer. An m-ary partition of a nonnegative integer n is a partition of n such that each part is a power of m. If there is "no gaps" in the parts, i.e., whenever m^i is a part, $1, m, m^2, \ldots, m^{i-1}$ are parts, then the partition is called a

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restricted m-ary partition. The number of restricted m-ary partitions of n is denoted by $c_m(n)$. Notice that the generating function of $c_m(n)$ is given by

$$C_m(q) := \sum_{n=0}^{\infty} c_m(n)q^n = 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+\cdots+m^n}}{(1-q)(1-q^m)\cdots(1-q^{m^n})}.$$

In recent years, the arithmetic properties of m-ary partitions and restricted m-ary partitions have been widely studied since Churchhouse [5] initiated the study of 2-ary partitions in the late 1960s. For example, Rødseth [7] extended Churchhouse's results to include p-ary partition functions $b_p(n)$, where p is any prime. Andrews [1], Gupta [6] and Rødseth and Sellers [8] studied further the congruences for $b_m(n)$, where $m \geq 2$ is any positive integer. And later, Andrews, Fraenkel and Sellers [2,3] provided characterizations of the values $b_m(mn)$ and $c_m(mn)$ modulo m. Andrews, Brietzke, Rødseth and Sellers [4] proved that for odd prime m,

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{m^j},$$

for all $n \geq 0$ and $0 \leq j < m$. In this note, we will show that

Theorem 1.1. *For a fixed integer $m \geq 2$ and for all nonnegative integer n, we have*

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{\frac{m^j}{c_j}},$$

where $c_j = 1$ if m is odd and $c_j = 2^{j-1}$ if m is even.

This confirms the conjecture of Andrews et al. [4, Conjecture 4.1].

2. Proof of Theorem 1.1

Our proof is based on the following results and notations of Andrews et al. [4]. Suppose that

$$\binom{km+1}{j} = \sum_{i=0}^{j} s_{j,i}\binom{k}{i},$$

where $s_{j,i}$ is an integer independent of k such that $s_{j,0} = 0$ if $j > 1$ and $s_{j,0} = 1$ otherwise. And denote

$$h_j = \frac{q^j}{(1-q)^{j+1}}.$$