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On quotients of values of Euler's function on the Catalan numbers

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ABSTRACT

In a recent work, Luca and Stănică examined quotients of the form $\frac{\varphi(C_m)}{\varphi(C_n)}$, where φ is Euler's totient function and $C_0, C_1, C_2 \dots$ is the sequence of the Catalan numbers. They observed that the number 4 (and analogously $\frac{1}{4}$) appears noticeably often as a value of these quotients. We give an explanation of this phenomenon, based on Dickson's conjecture. It turns out not only that the value 4 is (in a certain sense) special in relation to the quotients $\frac{\varphi(C_{n+1})}{\varphi(C_n)}$, but also that the value 4^k has similar "special" properties with respect to the quotients $\frac{\varphi(C_{n+k})}{\varphi(C_n)}$, and in particular we show that Dickson's conjecture implies that, for each k, the number 4^k appears infinitely often as a value of the quotients $\frac{\varphi(C_{n+k})}{\varphi(C_n)}$.

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1. Introduction

A well-known Carmichael's conjecture [4] states that for each positive integer n there is a different positive integer m such that $\varphi(m) = \varphi(n)$, where φ is Euler's totient function.

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Though there are some results related to this conjecture—for example, it is known that a counterexample, if exists, must be greater than $10^{10^{10}}$, and that in that case there are infinitely many counterexamples [8]; it is known that if there exists an integer n such that for each prime p for which $p-1 \mid \varphi(n)$ we have $p^2 \mid n$, then n is a counterexample to Carmichael's conjecture [13]; it is known that every positive integer greater than 1 is a multiplicity of some value of φ [9] etc.—but the solution to Carmichael's conjecture seems out of reach.

Instead of studying the equality of values of φ, this line of research can be generalized by considering the quotients of the values of φ. In a recent work [10], Luca and Stănică did this with a restriction of the domain to the Catalan numbers $C_0, C_1, C_2 \dots$; in other words, they examined quotients of the form $\frac{\varphi(C_m)}{\varphi(C_n)}$. They observed that the number 4 (and analogously $\frac{1}{4}$) appears noticeably often as a value of these quotients. In this work we give an explanation of this phenomenon, based on Dickson's conjecture. Furthermore, it turns out not only that the value 4 is (in a certain sense) special in relation to the quotients $\frac{\varphi(C_{n+1})}{\varphi(C_n)}$, but also that the value 4^k has similar "special" properties with respect to the quotients $\frac{\varphi(C_{n+k})}{\varphi(C_n)}$, and in particular we show that Dickson's conjecture implies that, for each k, the number 4^k appears infinitely often as a value of the quotients $\frac{\varphi(C_{n+k})}{\varphi(C_n)}$.

The paper is organized as follows. In Section 2 we state Dickson's conjecture and prove two lemmas that will be useful later. In Section 3 we consider the quotient $\frac{\varphi(C_{n+1})}{\varphi(C_n)}$, where we show that for all integers n of a certain, arguably quite general form, we have the equality $\frac{\varphi(C_{n+1})}{\varphi(C_n)} = 4$; results of Luca and Stănică on this topic are, as is to be shown, two special cases of this theorem. Section 4 is devoted to quotients of the form $\frac{\varphi(C_{n+k})}{\varphi(C_n)}$. Finally, in the Appendix A given at the end we collect some results, needed during the work, on divisibility of the Catalan numbers by primes and prime powers.

2. Preliminaries

Dickson's conjecture [5] generalizes Dirichlet's theorem on primes in arithmetic progressions and implies many interesting results such as the infinitude of twin primes, the infinitude of Sophie Germain primes, the infinitude of composite Mersenne numbers etc. (and, needless to say, its proof seems to be hopelessly beyond reach).

Dickson's Conjecture. Let $P_i(x) = a_i x + b_i$ for $1 \leqslant i \leqslant k$, where $a_i, b_i \in \mathbb{Z}$ and $a_i \geqslant 1$. Then there are infinitely many positive integers x such that $P_1(x), P_2(x), \dots, P_k(x)$ are all primes, unless there exists a prime number p such that for each $x \in \mathbb{Z}$ at least one of $P_1(x), P_2(x), \dots, P_k(x)$ is divisible by p.

The following lemma gives two sufficient conditions (which will be enough for our needs) for the conditions from Dickson's conjecture to be satisfied.

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